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On a problem of Gaschütz

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ABSTRACT

Recall that a *formation* is a class of finite groups (up to isomorphism) that is closed under taking quotient groups and (pairwise) subdirect products. For a group G let $\mathrm{form}(G)$ denote the smallest formation containing G. There was a conjecture (the Gaschütz problem) that $\mathrm{form}(G)$ contains only finitely many subformations, for any G. This was proved in the case where G is solvable, and in some other cases. In the article a counterexample is constructed. Namely, if $A = 2S_5$ (a double cover of S_5), then $\mathrm{form}(A)$ has infinitely many subformations. The precise structure of the lattice of subformations of $\mathrm{form}(A)$ is found. The main technical tool is a reduction to the problem of finding submodule structure for some module over a certain category.

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1. Introduction

All the groups considered in this article are finite.

Recall that a *formation* is a class of finite groups, up to isomorphism, that is closed under taking quotients and subdirect products. Formations were introduced by Gaschütz [1]. They are studied in books [2] and [3].

If G is a group and $N_1, N_2 \trianglelefteq G$, then $G/N_1 \cap N_2$ is a subdirect product of G/N_1 and G/N_2. It easily follows that for any group G and a formation $\mathfrak{F}$ there exists a normal subgroup $G^{\mathfrak{F}} \trianglelefteq G$ (which is called the $\mathfrak{F}$-*residual* of G) such that $G/N \in \mathfrak{F}$ if and only if $N \supseteq G^{\mathfrak{F}}$.

Example. If p is a prime and $\mathfrak{F}$ is the class of all p-groups, then $\mathfrak{F}$ is a formation and $G^{\mathfrak{F}} = O^p(G)$.

It is easy to see that for any set $\mathfrak{M}$ of groups there exists a formation $\mathrm{form}(\mathfrak{M})$, *generated* by $\mathfrak{M}$, i.e., the least formation containing $\mathfrak{M}$. When $\mathfrak{M} = \{G\}$ we write just $\mathrm{form}(G)$ (omitting braces).

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There is an open question regarding formations (see e.g. [2, Problem 2], or [3, §8.1], or [4, Problem 9.59]), sometimes called *the Gaschütz problem*.

Is it true, that for every finite group G the formation $\mathrm{form}(G)$ *contains only finitely many subformations?*

In the following cases the answer is affirmative:

(1) if G is solvable [5];
(2) if the solvable residual $G^{\mathfrak{S}}$ contains no Frattini chief factors of G ([6]; see [3] for definitions);
(3) if G is a central product of several quasisimple groups ([5] and [7]);
(4) if G is an extension of a solvable group by a nonabelian simple group [8];
(5) if the socle length $l_s(G) \leqslant 2$ [9]. (The *socle length* $l_s(G)$ is defined inductively by $l_s(1) = 0$ and $l_s(G) = 1 + l_s(G/\mathrm{Soc}(G))$ when $G \neq 1$.)

In general, however, the answer is negative.

Let S_n be the symmetric group of degree n. By a *group of type* $2S_n$ we mean a group G such that $Z(G) \cong Z_2$, $G/Z(G) \cong S_n$, and G is not isomorphic neither to $S_n \times Z_2$ nor the subdirect product of S_n and Z_4. It is well known [10, §6.7], that for $n \geqslant 5$ there are precisely two nonisomorphic groups of type $2S_n$.

Theorem 1.1. *Let A be a group of type $2S_5$ and let $\mathfrak{Q} = \mathrm{form}(A)$. Then $\mathfrak{Q}$ has an infinite number of subformations.*

Remark. It will be clear from the proof that the theorem remains true for a wider class of groups, for example for all $2S_n$ with $n \geqslant 7$. We consider only $2S_5$ because our aim is to find at least one counterexample.

Theorem 2.5 gives a detailed description of the lattice of subformations of $\mathfrak{Q}$.

The work is organized as follows. The description of subformations of $\mathfrak{Q}$ can be, in fact, divided into two independent parts, namely (a) a reduction theorem that reduces the description of subformations of $\mathfrak{Q}$ to a certain representation-theoretic problem, and (b) a solution of this representation problem. Section 2 contains the subdivision of the proof into these (a) and (b), while (a) and (b) themselves are contained in Sections 3 and 4, respectively. Finally, in Section 5 we describe counterexamples to some other “finiteness questions” for formations.

2. Modules over categories

Prerequisites and notation Despite of we study a problem on finite groups, we need some elementary tools from category theory. All necessary information may be found in [11, Ch. 1, §§1–3]; see also [12, Ch. 1, §§7–8].

On the other hand, we do not need any preliminaries on formations. We prove by ourselves the several elementary facts that will be needed.

By S_n and A_n we denote the symmetric and alternating groups, respectively. The remaining group theoretic notation ($Z(G)$, $O_p(G)$, $O^p(G)$, $\Phi(G)$, $[X,Y]$, Z_n, $GL(n,q)$, $SL(n,q)$, $GL(V)$, $SL(V)$, etc.) is standard and can be found in [13].

For categories we use standard notation (see [11]) $\mathrm{Ob}\,\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. By $\mathrm{Funct}(\mathcal{C},\mathcal{D})$ we denote the set of all (covariant) functors from $\mathcal{C}$ to $\mathcal{D}$. We sometimes use symbol $\rightsquigarrow$ for functors, like $V \rightsquigarrow V^*$, when the action on morphisms is clear from the context. Also, we denote by **fSet**, **fGrp**, and **fVect**$_k$ the categories of all finite sets, finite groups and finite dimensional vector spaces over a field k, respectively.

Modules over categories Let $\mathcal{C}$ be a category and k be a field. We are given a (finite dimensional) *$k\mathcal{C}$-module* M, if for every $X \in \mathrm{Ob}\,\mathcal{C}$ we are given a finite dimensional k-space $M(X)$ and for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ we have a linear map $M(f): M(X) \longrightarrow M(Y)$, and $M(\mathrm{id}_X) = \mathrm{id}_{M(X)}$ and

$M(gf) = M(g)M(f)$ for all $X, Y, Z \in \operatorname{Ob}\mathcal{C}$, $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$, $g \in \operatorname{Hom}_{\mathcal{C}}(Y, Z)$. In other words, a $k\mathcal{C}$-module is nothing else but a functor from $\mathcal{C}$ to $\mathbf{fVect}_k$. We often write f_* for $M(f)$. (In general, we denote by f_* and f^* the images of a morphism f under a covariant, respectively contravariant functor, when it is clear which functor is considered.)

Example. Let G be a group (or a monoid). Take symbols p and φ_g, $g \in G$, and consider the category $\mathcal{C}$ that has a unique object p and morphisms $\{\varphi_g \mid g \in G\}$, the morphisms being multiplied as $\varphi_g\varphi_h = \varphi_{gh}$ by definition. Then the $k\mathcal{C}$-modules are the usual kG-modules.

If for each $X \in \operatorname{Ob}\mathcal{C}$ we have a subspace $L(X) \subseteq M(X)$ and $M(f)L(X) \subseteq L(Y)$ for all $X, Y \in \operatorname{Ob}\mathcal{C}$, $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$, then we say that L is a *$k\mathcal{C}$-submodule* of M, and write $L \subseteq M$. Next, define inclusion of submodules in an obvious way: if K and L are submodules of a $k\mathcal{C}$-module M, then $K \subseteq L$ means that $K(X) \subseteq L(X)$ for every $X \in \operatorname{Ob}\mathcal{C}$.

Example. Let $\mathcal{C} = \mathbf{fVect}_k$ and let Q be the $k\mathcal{C}$-module that assigns to a space X its tensor square $Q(X) = X \otimes X$, and to a linear map $f : X \longrightarrow Y$ a linear map $Q(f) = f \otimes f : X \otimes X \longrightarrow Y \otimes Y$. For $X \in \mathcal{C}$ let $L(X)$ be the subspace of symmetric (with respect to the map $x \otimes y \longmapsto y \otimes x$) elements. Then L is a submodule of Q.

The majority of modules over categories in this article has the following very special property.

Definition. A $k\mathcal{C}$-module M is *epimorphic*, if $M(f) : M(X) \longrightarrow M(Y)$ is an epimorphism for all $X, Y \in \mathcal{C}$, $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$. A submodule $L \subseteq M$ is epimorphic, if always $M(f)L(X) = L(Y)$.

Let M be a $k\mathcal{C}$-module and $S_1, S_2 \subseteq M$ be its submodules. For $X \in \mathcal{C}$ we set $S(X) = S_1(X) + S_2(X)$. It is easy to see that $S : X \rightsquigarrow S(X)$ is a $k\mathcal{C}$-submodule in M. Denote $S = S_1 + S_2$. If S_1 and S_2 are epimorphic, so is S. Moreover, if $S_1 \subseteq S_2 \subseteq \cdots$ is a non-decreasing chain of epimorphic submodules, then $\bar{S}$ defined by $\bar{S}(X) = \bigcup_{i=1}^{\infty} S_i(X)$, $X \in \mathcal{C}$, is an epimorphic submodule in M.

The category $\mathcal{D}$ Let I be a finite set. By k^I we denote the coordinate space over k whose coordinates are indexed by I; that is, k^I is the set of all arrays $(\lambda_i \in k \mid i \in I)$. We also can identify k^I with $\operatorname{Fun}(I, k)$, the set of all functions from I to k.

If $f : I \longrightarrow J$ is a map of sets, then we have a map $f^* : k^J \longrightarrow k^I$. Namely, for $u = (u_j \mid j \in J)$ we define $f^*(u) = v = (v_i \mid i \in I)$, where $v_i = u_{f(i)}$. In terms of functions f^* is just a composition: $f^*(u) = u \circ f$. Clearly, $I \longmapsto k^I$, $f \longmapsto f^*$ is a contravariant functor from $\mathbf{fSet}$ to $\mathbf{fVect}_k$.

We call a subspace $V \subseteq k^I$ *wide*, if for every $i \in I$ there exists $v = (v_j \mid j \in I) \in V$ such that $v_i \neq 0$. When $I = \emptyset$ then 0 is wide by definition.

It is obvious that

2.1. *If V is wide, then $f^*(V)$ is also wide.*

Below we consider only spaces over $k = \mathbf{F}_2 = \{0, 1\}$, the field of two elements, unless otherwise stated.

Consider the following category $\mathcal{D}$. The objects of $\mathcal{D}$ are pairs (I, V), where $I \in \mathbf{fSet}$ and $V \subseteq k^I$ is wide. A morphism from $X = (I, V)$ to $Y = (J, U)$ is, by definition, a symbol μ_f, where $f : J \longrightarrow I$ is a set injection such that $f^*(V) = U$. Let $X = (I, V)$, $Y = (J, U)$, $Z = (K, W)$, and $\mu_a \in \operatorname{Hom}_{\mathcal{D}}(X, Y)$, $\mu_b \in \operatorname{Hom}_{\mathcal{D}}(Y, Z)$. Then the composition map $K \xrightarrow{b} J \xrightarrow{a} I$ is an injection, and $(ab)^*(V) = b^*a^*(V) = b^*(a^*(V)) = b^*(U) = W$, so $\mu_{ab} \in \operatorname{Hom}_{\mathcal{D}}(X, Z)$. We define $\mu_b\mu_a = \mu_{ab}$. Also define $\mathrm{id}_X = \mu_{\mathrm{id}_I}$. It is easy to check that these data satisfy the axioms of a category.

The following statement is left as an exercise for the reader in order to get used to category $\mathcal{D}$.

2.2. *All morphisms in $\mathcal{D}$ are epimorphisms. Any endomorphism is an automorphism.*

(Recall that a morphism $f : X \longrightarrow Y$ in arbitrary category $\mathcal{C}$ is called an epimorphism if the map $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z)$, defined by $h \longmapsto hf$, is injective, for each $Z \in \mathrm{Ob}\,\mathcal{C}$.)

Define a $k\mathcal{D}$-module, denoted by H. For $X = (I, V) \in \mathcal{D}$ put $H(X) = k^I$, and for $X = (I, V)$, $Y = (J, U)$, $\varphi = \mu_f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ put $\varphi_* = H(\varphi) = f^*$. Note that D_1, defined by $D_1((I, V)) = V$, is an epimorphic submodule in H; and H itself is epimorphic also.

Theorem 2.3. *All distinct epimorphic submodules of H are some submodules, denoted by D_i, E_i, $i = 0, 1, \ldots$, $\overline{D}$, and H. These submodules satisfy the conditions: $D_0 = 0$, $D_i \subset D_j$ when $i < j$, $\overline{D} = \bigcup_{i=0}^{\infty} D_i$, $E_i = D_i + E_0$, $E_0 \subset \overline{D}$. All the inclusions of these submodules are: $H \supset \overline{D}$; $\overline{D} \supset D_i, E_i$ for all i; $D_i \subset D_j$ and $E_i \subset E_j$ when $i < j$; and $D_i \subset E_j$ when $i \leqslant j$.*

This theorem will be proved in Section 4.

Subformations of $\mathfrak{Q}$ Throughout the article $\mathfrak{I} = \mathrm{form}(\{1\})$ is the formation of trivial groups, $\mathfrak{E} = \mathrm{form}(Z_2)$ is the formation of all elementary abelian 2-groups, $\mathfrak{P} = \mathrm{form}(S_5)$ is the formation generated by S_5, and $\emptyset$ is the empty formation. Obviously $\emptyset \subset \mathfrak{I} \subset \mathfrak{E} \subset \mathfrak{P} \subseteq \mathfrak{Q}$.

Theorem 2.4. *Let F be the set of all subformations of $\mathfrak{Q}$ distinct from $\emptyset$, $\mathfrak{I}$, and $\mathfrak{E}$. Then $\mathfrak{P} \subseteq \mathfrak{F}$ for any $\mathfrak{F} \in F$. Next, let S be the set of all epimorphic $k\mathcal{D}$-submodules of H. Then there exists a bijection $\varkappa : F \longrightarrow S$ that inverts inclusions (i.e. $\varkappa(\mathfrak{X}) \supset \varkappa(\mathfrak{Y})$ if and only if $\mathfrak{X} \subset \mathfrak{Y}$).*

This theorem will be proved in Section 3.

Theorem 2.5. *Let A be a group of type $2S_5$, and $\mathfrak{Q} = \mathrm{form}(A)$. Then all the distinct subformations in $\mathfrak{Q}$ are $\emptyset$, $\mathfrak{I}$, $\mathfrak{E}$, $\mathfrak{P}$, and some formations denoted by $\overline{\mathfrak{K}}$, $\mathfrak{K}_i$ and $\mathfrak{H}_i$, $i = 0, 1, \ldots$.*

All the inclusions of these subformations are: $\emptyset \subset \mathfrak{I} \subset \mathfrak{E} \subset \mathfrak{P} \subset \overline{\mathfrak{K}}$; $\overline{\mathfrak{K}}$ is contained in all $\mathfrak{K}_i$ and $\mathfrak{H}_i$; $\mathfrak{K}_i \supset \mathfrak{K}_j$ and $\mathfrak{H}_i \supset \mathfrak{H}_j$ when $i < j$, and $\mathfrak{K}_i \supset \mathfrak{H}_j$ if $i \leqslant j$. Finally, $\mathfrak{K}_0 = \mathfrak{Q}$, and $\overline{\mathfrak{K}} = \bigcap_{i=0}^{\infty} \mathfrak{K}_i = \bigcap_{i=0}^{\infty} \mathfrak{H}_i$.

Proof. Let $\varkappa : F \longrightarrow S$ be as in Theorem 2.4. Set $\overline{\mathfrak{P}} = \varkappa^{-1}(H)$, $\overline{\mathfrak{K}} = \varkappa^{-1}(\overline{D})$, $\mathfrak{K}_i = \varkappa^{-1}(D_i)$, and $\mathfrak{H}_i = \varkappa^{-1}(E_i)$. Since H is the biggest element of S, it follows that $\overline{\mathfrak{P}}$ is the least element of F, i.e., $\overline{\mathfrak{P}} = \mathfrak{P}$. Thus, the first assertion is proved. The second one follows immediately from the fact that $\varkappa$ inverts inclusions.

The last assertion is a straightforward corollary of the previous ones (for example, $\mathfrak{K}_0$ contains all the remaining subformations, so $\mathfrak{K}_0$ is nothing else but $\mathfrak{Q}$). □

The categories of $k\mathcal{C}$-modules In this paragraph k is an arbitrary field.

Let $\mathcal{C}$ be a category and $k\mathcal{C}$-**Mod** be the totality of all $k\mathcal{C}$-modules. Then $k\mathcal{C}$-**Mod** is itself a category. Let K, L be $k\mathcal{C}$-modules. Then a (homo)morphism $h : K \longrightarrow L$ is, by definition, a collection of linear maps

$$h = \{h_X : K(X) \longrightarrow L(X) \mid X \in \mathcal{C}\}$$

such that for all $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the diagram

$$\begin{tikzcd} K(X) \arrow[r, "K(f)"] \arrow[d, "h_X"'] & K(Y) \arrow[d, "h_Y"] \\ L(X) \arrow[r, "L(f)"] & L(Y) \end{tikzcd}$$

commutes. In other words, h is a natural transformation (a functor morphism) of the functor M to L. If $h : K \longrightarrow L$ and $g : L \longrightarrow M$ are morphisms of $k\mathcal{C}$-modules, then their product is defined by $(gh)_X = g_X h_X$ for every $X \in \mathrm{Ob}\,\mathcal{C}$. It is easy to see that $k\mathcal{C}$-**Mod** is a category under this multiplication of morphisms. This is a particular case of the fact that for arbitrary categories $\mathcal{C}$ and $\mathcal{C}'$ the set of functors $\mathrm{Funct}(\mathcal{C},\mathcal{C}')$ is a category whose morphisms are functor morphisms. For example, if $\mathcal{C}$ is the unique object category associated to a group (or a monoid) G, then the category of $k\mathcal{C}$-modules is the usual category of kG-modules.

It is clear that if $f : M \longrightarrow N$ is a $k\mathcal{C}$-module isomorphism, then f induces a bijection of sets of submodules of M and N; this bijection preserves inclusions and takes epimorphic submodules to epimorphic ones.

2.6. *If $\mathcal{C}$, $\mathcal{C}'$ are categories, $F_1, F_2 \in \mathrm{Funct}(\mathcal{C},\mathcal{C}')$, and $\theta : F_1 \longrightarrow F_2$ is a natural transformation, then θ is a functor isomorphism if and only if $\theta_X : F_1(X) \longrightarrow F_2(X)$ is a $\mathcal{C}'$-isomorphism for every $X \in \mathrm{Ob}\,\mathcal{C}$.*

Proof. See [11, Proposition 1.5]. □

Hence we obtain the following criterion for two $k\mathcal{C}$-modules to be isomorphic.

2.7. *Let $h : L \longrightarrow M$ be a morphism of $k\mathcal{C}$-modules. Then h is an isomorphism of $k\mathcal{C}$-modules if and only if $h_X : L(X) \longrightarrow M(X)$ is an isomorphism of k-spaces for each $X \in \mathrm{Ob}\,\mathcal{C}$.*

Let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcategory; then we may consider $L' = \{L(X) \mid X \in \mathrm{Ob}\,\mathcal{C}'\}$ as a $k\mathcal{C}'$-module. The action of $\mathcal{C}'$-morphisms on $L(X)$ is the same as that in $\mathcal{C}$. We call L' the *restriction* of L to $\mathcal{C}'$ and write $L' = L|_{\mathcal{C}'}$.

Remark. The notion of a module over a category is rather natural; but it seems to be not a well-known one. However, it is not new; see, e.g., [14], where the equivalent concept of linear representation of a category was considered. Another related concept is the concept of a polynomial functor, see [15].

3. Subdirect products and formations

Maps of direct products Let $\{G_i \mid i \in I\}$ be a finite family of finite groups. Denote their external direct product $\prod_{i\in I} G_i$ by G_I.

We write the elements of G_I in the form $g = (g_i \mid i \in I)$, where $g_i \in G_i$. For $i \in I$ let $\pi_i : G \longrightarrow G_i$ be the projection of G_I onto G_i, and $\rho_i : G_i \longrightarrow G_I$ be the obvious embedding. Further, let $J \subseteq I$ be any subset and $G_J = \prod_{i\in J} G_i$. Then we have an obvious projection $\pi_J : G_I \longrightarrow G_J$ and embedding $\rho_J : G_J \longrightarrow G_I$.

In the case all G_i are isomorphic copies of a group G we write G^I instead of G_I. (Strictly speaking, the phrase "isomorphic copies of G" means that we have a collection of groups $\{G_i \mid i \in I\}$ together with a collection of isomorphisms $\gamma_i : G \longrightarrow G_i$. We imagine these G_i as being identified with G by means of γ_i and, as a rule, do not mention γ_i explicitly.)

For an arbitrary map of sets $f : I \longrightarrow J$ define a homomorphism $f^* = f^*_G : G^J \longrightarrow G^I$ by $f^*((g_j \mid j \in J)) = (h_i \mid i \in I)$, where $h_i = g_{f(i)}$. If f is injective (surjective), then f^* is surjective (respectively, injective). It is easy to see that $(gf)^*_G = f^*_G g^*_G$ for arbitrary $f : I \longrightarrow J$, $g : J \longrightarrow K$, and G. Next, for a homomorphism $\varphi : G \longrightarrow H$ we define a homomorphism $\varphi_I : G^I \longrightarrow H^I$ by $\varphi_I((g_i \mid i \in I)) = (\varphi(g_i) \mid i \in I)$. Evidently, $(\psi\varphi)_I = \psi_I\varphi_I$ for any set I and homomorphisms $\varphi : G \longrightarrow H$, $\psi : H \longrightarrow L$.

3.1. *Let $f : I \longrightarrow J$ be an arbitrary map, and $\varphi : G \longrightarrow H$ be a homomorphism. Then the diagram*

$$\begin{tikzcd} G^J \arrow[r, "\varphi_J"] \arrow[d, "f^*_G"'] & H^J \arrow[d, "f^*_H"] \\ G^I \arrow[r, "\varphi_I"] & H^I \end{tikzcd}$$

is commutative.

The proof, as well as the proof of Proposition 3.2 below, is straightforward and left to the reader.

We will consider various "componentwise" maps of direct products, that are more general than f_G^* or φ_I. A typical example is the following. Let G_1, G_2, G_3, H_1, H_2, and H_3 be groups, and $\alpha : G_1 \longrightarrow H_2$, $\beta : G_3 \longrightarrow H_1$, and $\gamma : G_3 \longrightarrow H_3$ be homomorphisms. Then we may consider a map (which is a homomorphism) $\zeta : G_{\{1,2,3\}} \longrightarrow H_{\{1,2,3\}}$ defined by

$$\zeta\big((g_1, g_2, g_3)\big) = \big(\beta(g_3), \alpha(g_1), \gamma(g_3)\big).$$

It is convenient to collect all the maps of such kind to form a category. Namely, let $\mathcal{P}$ be the category whose objects are the pairs $(I, \{G_i \mid i \in I\})$, where I is a finite set and $\{G_i \mid i \in I\}$ is a collection of finite groups indexed by I. A morphism from $(I, \{G_i \mid i \in I\})$ to $(J, \{H_j \mid j \in J\})$ is a pair (f, φ), where $f : J \longrightarrow I$ is a map of sets and φ is an array $\varphi = (\varphi_j \mid j \in J)$, where $\varphi_j : G_{f(j)} \longrightarrow H_j$ are group homomorphisms. (Note that f and g act, in a sense, in opposite directions.) If $X = (I, \{G_i \mid i \in I\})$, $Y = (J, \{H_j \mid j \in J\})$, and $Z = (K, \{L_k \mid k \in K\})$ are three objects of $\mathcal{P}$, and $(f, \varphi) : X \longrightarrow Y$ and $(g, \psi) : Y \longrightarrow Z$ are $\mathcal{P}$-morphisms, then we define their product by $(g, \psi)(f, \varphi) = (fg, \chi)$, where $\chi = (\chi_k \mid k \in K)$, $\chi_k = \psi_k \varphi_{g(k)} : G_{fg(k)} \longrightarrow H_{g(k)} \longrightarrow L_k$.

Next, define a map $\prod : \mathcal{P} \longrightarrow \mathbf{fGrp}$ that takes an object $(I, \{G_i \mid i \in I\})$ to $G_I = \prod_{i \in I} G_i$ and a morphism $(f, \varphi) : (I, \{G_i \mid i \in I\}) \longrightarrow (J, \{H_j \mid j \in J\})$ to a group homomorphism $(f, \varphi)_* : G_I \longrightarrow H_J$, that sends $(g_i \mid i \in I)$ to $(h_j \mid j \in J)$, where h_j are defined by $h_j = \varphi_j(g_{f(j)})$.

Proposition 3.2. *$\mathcal{P}$ is a category, and $\prod : \mathcal{P} \longrightarrow$ **fGrp** is a functor.*

The structure of subdirect product Let $G_I = \prod_{i \in I} G_i$ and $G \subseteq G_I$. We say that G is *wide*, if $\pi_i(G) = G_i$ for all $i \in I$. Next, we say that a group X is a *subdirect product* of a family $\{X_i \mid i \in I\}$ if X is isomorphic to a wide subgroup of X_I.

3.3. *Let $X, Y \in \mathcal{P}$, $X = (I, \{G_i \mid i \in I\})$, $Y = (J, \{H_j \mid j \in J\})$, $\zeta = (f, \varphi) \in \mathrm{Hom}_{\mathcal{P}}(X, Y)$, $\varphi = (\varphi_j \mid j \in J)$, $\varphi_j : G_{f(j)} \longrightarrow H_j$, and let $\prod(\zeta) = \zeta_* : G_I \longrightarrow H_J$ be the corresponding group homomorphism. Suppose that φ_j is an epimorphism for all j. Then*

1) *if a subgroup $G \subseteq G_I$ is wide, then $\zeta_*(G) \subseteq H_J$ is wide*;
2) *if f is injective and $H \subseteq H_J$ is wide, then $\zeta_*^{-1}(H) \subseteq G_I$ is wide also.*

Proof. 1) Let $x \in H_j$. Take an element $y \in G_{f(j)}$ such that $\varphi_j(y) = x$. Since G is wide, there exists $g \in G$ such that $\pi_{f(j)}(g) = y$. Then $\zeta_*(g)$ is in $\zeta_*(G)$ and its j-th coordinate is x. So $\zeta_*(G)$ is wide.

2) Let $x \in G_i$. We need to find an element $g \in G_I$ such that $\zeta_*(g) \in H$ and the i-th coordinate of g equals x.

First assume $i \notin f(J)$. Take $g = \rho_i(x)$. Then $\zeta_*(g) = 1$ and $\pi_i(g) = x$.

Next suppose $i = f(j)$, $j \in J$. Let $y = \varphi_j(x)$. Take $h \in H$ such that $h_j = \pi_j(h) = y$. Next, take elements $g_l \in G_l$, $l \in I$ in the following way. For $l = i$ put $g_l = x$; for $l = f(t)$, $t \neq j$ take for g_l any element such that $\varphi_t(g_l) = h_t$; and for $l \notin f(J)$ choose g_l in an arbitrary way. Let $g = (g_l \mid l \in I)$. Clearly, g is the desired element. □

Corollary 3.4.

1) *Let $G_I = \prod_{i \in I} G_i$ and $J \subseteq I$. Then*
   (a) *if $G \subseteq G_I$ is wide, then $\pi_J(G)$ is wide*;
   (b) *if $H \subseteq G_J$ is wide, then $\pi_J^{-1}(H)$ is wide.*
2) *Let $G_I = \prod_{i \in I} G_i$ and $H_I = \prod_{i \in I} H_i$, and suppose $\varphi_i : G_i \longrightarrow H_i$ are epimorphisms for all i. Then*
   (a) *if $G \subseteq G_I$ is wide, then $\varphi_I(G) \subseteq H_I$ is wide*;
   (b) *if $H \subseteq H_I$ is wide, then $\varphi_I^{-1}(H) \subseteq G_I$ is wide.*

Proof. These are particular cases of statement 3.3. □

Suppose $G \subseteq G_I$ is wide, and let $J \subseteq I$. Then $\pi_J(G)$ is a wide subgroup of G_J. We say that G is *reduced* if $\pi_J(G)$ is a proper quotient of G for every proper subset $J \subset I$.

3.5. *Let $G \subseteq G_I$ be a wide subgroup. Then there exists a subset $J \subseteq I$ such that G is isomorphic to a reduced wide subgroup of G_J. More precisely, let $J \subseteq I$ be a minimal subset such that $\pi_J(G) \cong G$. Then $\pi_J(G) \subseteq G_J$ is reduced.*

Proof. Let $K \subset J$ be a proper subset. Then $\pi_K(\pi_J(G)) = \pi_K(G)$ is a proper quotient of G and thereby of $\pi_J(G)$. $\square$

Now we introduce a useful notational convention. Let $G_I = \prod_{i \in I} G_i$, let $J \subseteq I$ be a subset, and for each $j \in J$ let $H_j \subseteq G_j$ be a subgroup. Then we denote by $\widehat{H}_J$ the image of the composition map $H_J \longrightarrow G_J \xrightarrow{\rho_J} G_I$, i.e., $\widehat{H}_J = \langle \rho_j(H_j) \mid j \in J \rangle$. Note that according to this definition $\widehat{H}_{\{i\}} = \rho_i(H_i)$, which we also denote by $\widehat{H}_i$. Similarly, suppose all G_i are isomorphic copies of a group G, that is identified with G_i via $\gamma_i : G \longrightarrow G_i$, and let $H \subseteq G$ be a subgroup. Put $H_j = \gamma_j(H)$, $j \in J$. Then we denote $\widehat{H}^J = \langle \rho_j(H_j) \mid j \in J \rangle$. The latter subgroup is isomorphic to H^J in the obvious way.

Recall that a group is *monolithic* if it has a unique minimal normal subgroup (the *monolith*).

3.6. *Suppose G_i, $i \in I$, are monolithic groups and $M_i \trianglelefteq G_i$ are their monoliths. Let $\overline{G}_i = G_i/M_i$ and let $\varphi_i : G_i \longrightarrow \overline{G}_i$ be the natural epimorphism, and $\varphi_I : G_I \longrightarrow \overline{G}_I = \prod_{i \in I} \overline{G}_i$ be the corresponding homomorphism of products. Then the reduced wide subgroups $G \subseteq G_I$ are precisely the groups of the form $\varphi_I^{-1}(H)$, where $H \subseteq \overline{G}_I$ is a wide subgroup (possibly not reduced).*

Proof. Clearly φ_I is an epimorphism and $\operatorname{Ker} \varphi_I = \widehat{M}_I$.

Show that if a subgroup $G \subseteq G_I$ is wide and reduced, then $\widehat{M}_I \subseteq G$. Let $i \in I$ and $T_i = \{x \in G_i \mid \rho_i(x) \in G\}$. Then $T_i \cong \widehat{T}_i = G \cap \widehat{G}_i = G \cap \operatorname{Ker} \pi_{I \setminus \{i\}}$, which is $\neq 1$ since G is reduced. Show that $T_i \trianglelefteq G_i$. Note that if $g \in G_i$ and $h = (h_j \mid j \in I) \in G_I$ are arbitrary elements, then $h\rho_i(g)h^{-1} = \rho_i(h_i g h_i^{-1})$. Now, let $x \in T_i$ and $y \in G_i$ be arbitrary elements and let $h \in G$ be an element such that $\pi_i(h) = y$. Then G contains $h\rho_i(x)h^{-1} = \rho_i(yxy^{-1})$, whence $yxy^{-1} \in T_i$. So T_i is normal in G_i. Since $T_i \neq 1$ and G_i is monolithic, it follows that $M_i \subseteq T_i$, whence $\widehat{M}_i \subseteq \widehat{T}_i \subseteq G$. So G contains $\langle \widehat{M}_i \mid i \in I \rangle = \widehat{M}_I$, as required.

As φ_I is an epimorphism and $G \supseteq \operatorname{Ker} \varphi_I$, we have $G = \varphi_I^{-1}(H)$, where $H = \varphi_I(G)$. By Corollary 3.4, 2)(a) H is a wide subgroup of $\overline{G}_I$.

Conversely, suppose $H \subseteq \overline{G}_I$ is wide, and let $G = \varphi_I^{-1}(H)$. Then G is a wide subgroup of G_I by Corollary 3.4, 2)(b). Also $G \supseteq \widehat{M}_i$ for any i, whence $G \cap \operatorname{Ker} \pi_{I \setminus \{i\}} \neq 1$, so G is reduced. $\square$

The construction of subdirect product is "associative" in the following sense.

3.7. *Let I and J be disjoint finite sets and $\{G_i \mid i \in I \cup J\}$ be a collection of groups. The following statements are equivalent:*

(i) *G is a subdirect product of the family $\{G_i \mid i \in I \cup J\}$;*
(ii) *there exist H_1 and H_2 such that H_1 is a subdirect product of $\{G_i \mid i \in I\}$, H_2 is a subdirect product of $\{G_i \mid i \in J\}$, and G is a subdirect product of H_1 and H_2.*

Proof. (i) $\Rightarrow$ (ii). We may assume G is a wide subgroup of $G_{I \cup J}$. Put $H_1 = \pi_I(G)$ and $H_2 = \pi_J(G)$. Then $H_1 \subseteq G_I$ and $H_2 \subseteq G_J$ are wide. Further, $H_1 \times H_2$ can be identified with a subgroup of $G_{I \cup J}$, and $G \subseteq H_1 \times H_2$. It is clear that $\pi_{H_1}(G) = H_1$, and similarly for H_2. That is, G is a wide subgroup of $H_1 \times H_2$.

(ii) $\Rightarrow$ (i). We may consider G as identified with a wide subgroup in $H_1 \times H_2$. Next, H_1 and H_2 can be identified with wide subgroups of G_I and G_J, respectively. So $H_1 \times H_2$ may be considered as embedded into $G_{I \cup J}$, and now G is identified with a subgroup of $G_{I \cup J}$. It is clear that $\rho_I(G) = H_1$, $\rho_J(G) = H_2$.

It remains to show that $G \subseteq G_{I\cup J}$ is wide. Let $\pi_i : G_{I\cup J} \longrightarrow G_i$ and $\pi_i' : G_I \longrightarrow G_i$ be projections, then $\pi_i = \pi_i' \pi_I$. Hence $\pi_i(G) = \pi_i'(\pi_I(G)) = \pi_i'(H_1) = G_i$. Similarly $\pi_j(G) = G_j$ for all $j \in J$. $\square$

Corollary 3.8. *A class $\mathfrak{F}$ of groups is closed under finite subdirect products if and only if $\mathfrak{F}$ is closed under taking pairwise subdirect products.*

Elementary facts on formations

Lemma 3.9. *Let $N_i \trianglelefteq G_i$, $i = 1, 2$, and let H be a subdirect product of groups G_1/N_1 and G_2/N_2. Then H is isomorphic to a group of the form G/N, where G is a subdirect product of G_1 and G_2, and $N \trianglelefteq G$.*

Proof. We may assume that H is a wide subgroup of $G_1/N_1 \times G_2/N_2$. Let G be the preimage of H under epimorphism $G_1 \times G_2 \longrightarrow G_1/N_1 \times G_2/N_2$. Then G is wide by Corollary 3.4. Clearly $H \cong G/N$, where $N = \widehat{N}_1 \times \widehat{N}_2$ is the kernel of the epimorphism. $\square$

3.10. *Let $\mathfrak{X}$ be a set of groups. Then* form$(\mathfrak{X})$ *consists of all groups of the form G/N, where G is a subdirect product of several groups isomorphic to some groups of $\mathfrak{X}$ and $N \trianglelefteq G$.*

Proof. Let $\mathfrak{Y}$ be the class of all groups of the described form. By 3.7 a subdirect product of several groups of $\mathfrak{X}$ can be obtained by composition of pairwise subdirect products. Now it follows from the definition of a formation that $\mathfrak{Y} \subseteq \mathrm{form}(\mathfrak{X})$. Obviously, $\mathfrak{Y}$ is closed under taking quotients. Prove that $\mathfrak{Y}$ is closed under subdirect products. Let $H_1, H_2 \in \mathfrak{Y}$ and let H be their subdirect product. We have $H_1 \cong G_1/N_1$ and $H_2 \cong G_2/N_2$, where G_i are subdirect products of several groups of $\mathfrak{X}$ and $N_i \trianglelefteq G_i$. By Lemma 3.9 we have $H \cong G/N$, where G is a subdirect product of G_1 and G_2. As both G_1 and G_2 are subdirect products of several groups of $\mathfrak{X}$, so is G by 3.7. So $H \in \mathfrak{Y}$.

Since $\mathfrak{Y}$ is closed under quotients and subdirect products, it follows that $\mathfrak{Y}$ is a formation. As $\mathfrak{X} \subseteq \mathfrak{Y}$, we have $\mathrm{form}(\mathfrak{X}) \subseteq \mathfrak{Y}$. Thus $\mathfrak{Y} = \mathrm{form}(\mathfrak{X})$. $\square$

Lemma 3.11. *Let $\varphi : G \longrightarrow H$ be a group epimorphism and $N \trianglelefteq G$. Then $\varphi(N) \trianglelefteq H$, and $H/\varphi(N)$ is a quotient of G/N.*

3.12. *If $\mathfrak{F}$ is any formation and $\varphi : G \longrightarrow H$ is an epimorphism, then $\varphi(G^{\mathfrak{F}}) = H^{\mathfrak{F}}$.*

Proof. Consider the composition map $G \xrightarrow{\varphi} H \longrightarrow H/H^{\mathfrak{F}}$. This is an epimorphism, and its kernel, which we denote by N, equals $\varphi^{-1}(H^{\mathfrak{F}})$. The image of this map is a group of $\mathfrak{F}$, so $G/N \in \mathfrak{F}$, whence $N \supseteq G^{\mathfrak{F}}$. Hence $\varphi^{-1}(H^{\mathfrak{F}}) \supseteq G^{\mathfrak{F}}$, so $\varphi(G^{\mathfrak{F}}) \subseteq H^{\mathfrak{F}}$. On the other hand, $H/\varphi(G^{\mathfrak{F}})$ is a quotient of $G/G^{\mathfrak{F}}$ and so is in $\mathfrak{F}$. So $\varphi(G^{\mathfrak{F}}) \supseteq H^{\mathfrak{F}}$. $\square$

Some notation Now we begin to consider the particular groups that we need. In the rest of this section we use the following notation.

$k = \mathbf{F}_2 = \{0, 1\}$ be the field of two elements, which we also consider as a group isomorphic to Z_2;
P be a group isomorphic to the symmetric group S_5, and $N = [P, P]$ be its normal subgroup of index 2;
$\sigma : P \longrightarrow k$ be (the unique) epimorphism;
Q be a group isomorphic to the group A of the Introduction;
$K = Z(Q)$, and $L = [Q, Q]$.

It will be sometimes convenient to distinguish between individual groups and their isomorphism classes. We use symbols P and Q for individual groups, and S_5 and A for their isomorphism classes. So we write $X \cong S_5$, but $\varphi : Y \longrightarrow Q$.

It follows from the well-known properties of alternating groups and their extensions (see [10]) that $1 \subset K \subset L \subset Q$ is the unique composition series of Q, and $K \cong Z_2$, $L/K \cong A_5$, $Q/L \cong Z_2$, $L \cong SL(2, 5)$, and $Q/K \cong S_5$. Further,

$\tau : Q \longrightarrow P$ be (a fixed) epimorphism;
$\widehat{\sigma} = \sigma\tau$; obviously, this is (the unique) epimorphism $Q \longrightarrow k$;
$\nu : k \longrightarrow Q$ be (the unique) embedding whose image is K;
I be a finite set;
P_i, Q_i, where $i \in I$, are isomorphic copies of the groups P and Q, $N_i \subset P_i$ and $K_i \subset L_i \subset Q_i$ are the subgroups corresponding to $N \subset P$ and $K \subset L \subset Q$, and $P^I = \prod_{i\in I} P_i$ and $Q^I = \prod_{i \in I} Q_i$ be the corresponding direct powers.

Next, the meaning of symbols $\widehat{N}_i, \widehat{N}^I \subseteq P^I$; $\widehat{Q}_i$, $\widehat{K}_i$, $\widehat{L}_i$, $\widehat{K}^I$, $\widehat{L}^I \subseteq Q^I$, as well as $\tau_I : Q^I \longrightarrow P^I$, $\widehat{\sigma}_I : Q^I \longrightarrow k^I$, and $\nu_I : k^I \longrightarrow Q^I$ must be clear from the previous discussion.

Finally, by V and W we usually denote some subspaces of k^I. Also we denote $P(I, V) = \sigma_I^{-1}(V) \subseteq P^I$ and $Q(I, V) = \widehat{\sigma}_I^{-1}(V) \subseteq Q^I$.

All the remaining notation will be introduced where it is used.

Subdirect products of several S_5

3.13. *The reduced wide subgroups of P^I are precisely the subgroups $P(I, V)$, where $V \subseteq k^I$ is a wide subspace.*

Proof. This immediately follows from 3.6 if we take $G_i = P_i$, $M_i = N_i$, and $\overline{G}_i = k$. □

3.14. *Suppose $V \subseteq k^I$ is wide, let $X = P(I, V)$, and let $C_i = X \cap \operatorname{Ker}\pi_i = \operatorname{Ker}\widehat{\pi}_i$, where $\widehat{\pi}_i = \pi_i|_X$, be the kernel of the projection of X onto P_i. Then $X/C_i \cong S_5$ and $C_i \neq C_j$ when $i \neq j$, and if $X/Y \cong S_5$, then $Y = C_i$ for appropriate i.*

Proof. It is clear that $X/C_i \cong S_5$. Obviously, $\widehat{N}_i \trianglelefteq X$. Next, $\pi_i(\widehat{N}_i) = N_i$ and $\pi_i(\widehat{N}_j) = 1$ when $j \neq i$. So $\widehat{N}_i \nsubseteq C_i$ and $\widehat{N}_i \subseteq C_j$, whence $C_i \neq C_j$.

Suppose $X/Y \cong S_5$ and $\varphi : X \longrightarrow P$ is an epimorphism whose kernel is Y. Then, for any i, the group $\varphi(\widehat{N}_i)$ is a normal subgroup of P and a homomorphic image of $\widehat{N}_i \cong A_5$, and so $\varphi(\widehat{N}_i) = 1$ or N.

Assume $\varphi(\widehat{N}_i) = 1$ for all i. Then $\widehat{N}^I \subseteq \operatorname{Ker}\varphi = Y$, so $X/Y \cong S_5$ is a homomorphic image of $X/\widehat{N}^I \cong V$, a contradiction. Next assume there are $i \neq j$ such that $\varphi(\widehat{N}_i) = \varphi(\widehat{N}_j) = N$. But $\widehat{N}_i$ and $\widehat{N}_j$ commute elementwise, so $\varphi(\widehat{N}_i)$ and $\varphi(\widehat{N}_j)$ must commute elementwise also, a contradiction. Thus, $\varphi(\widehat{N}_i) = N$ for a unique i.

Let $i \in I$ be such that $\varphi(\widehat{N}_i) = N$. Suppose $x \in X$ commutes with $\widehat{N}_i$. Then $\varphi(x)$ and $\varphi(\widehat{N}_i)$ commute, whence $\varphi(x) = 1$. Hence $C_X(\widehat{N}_i) \subseteq Y$. However, an element of X commutes with $\widehat{N}_i$ if and only if its i-th coordinate equals 1, so $C_X(\widehat{N}_i) = C_i$ and $C_i \subseteq Y$. Since $X/C_i \cong S_5$, it follows that $Y = C_i$. □

Now we can describe all (not necessary reduced) wide subgroups of P^I. We need an obvious observation.

Lemma 3.15. *Let G and H be any groups and $\varphi_1, \varphi_2 : G \longrightarrow H$ be some epimorphisms such that* $\operatorname{Ker}\varphi_1 = \operatorname{Ker}\varphi_2$. *Then there exists an automorphism $\psi \in \operatorname{Aut}(H)$ such that $\varphi_2 = \psi\varphi_1$.*

Let I be a finite set, $J \subseteq I$, $V \subseteq k^J$ be a wide subspace, $\alpha : I \setminus J \longrightarrow J$ be a map, and $\beta = (\beta_i \in \operatorname{Aut}(P) \mid i \in I \setminus J)$ be an array of elements of $\operatorname{Aut}(P)$ indexed by $I \setminus J$. Let $P(I, J, V, \alpha, \beta) \subseteq P^I$ be the subset of all elements $\bar{g} = (g_i \mid i \in I)$ such that $\pi_J(\bar{g}) \in P(J, V)$ and $g_i = \beta_i(g_{\alpha(i)})$ for all $i \in I \setminus J$.

3.16. *A subgroup $G \subseteq P^I$ is wide $\Leftrightarrow$ $G = P(I, J, V, \alpha, \beta)$ for some J, V, α, β.*

Proof. $\Leftarrow$) Let $X = (J, \{P_j \mid j \in J\})$ and $Y = (I, \{P_i \mid i \in I\}) \in \mathcal{P}$, and let $\zeta \in \operatorname{Hom}_{\mathcal{P}}(X, Y)$ be $\zeta = (f, \gamma)$, where $f : I \longrightarrow J$ is defined by $f(j) = j$ for $j \in J$ and $f(i) = \alpha(i)$ for $i \in I \setminus J$, and $\gamma = (\gamma_i \mid i \in I)$, where $\gamma_j = \mathrm{id}_P$ for $j \in J$ and $\gamma_i = \beta_i$ for $i \in I \setminus J$. It is easy to see that $P(I, J, V, \alpha, \beta)$ is nothing else but $\zeta_*(P(J, V))$ and is therefore a wide subgroup of P^I, by 3.3.

$\Rightarrow$) Take a minimal subset $J \subseteq I$ such that $G \cap \operatorname{Ker} \pi_J = 1$. Then π_J maps G isomorphically onto $\widehat{G} = \pi_J(G)$, and $\widehat{G} \subseteq P^J$ is a reduced wide subgroup. By 3.13 we obtain $\widehat{G} = P(J, V)$, where $V \subseteq k^J$ is a wide subspace.

For $j \in J$ let $C_j = \widehat{G} \cap \operatorname{Ker} \pi_j$. Let $\zeta = \pi_J|_G : G \longrightarrow \widehat{G}$, and let $\xi : \widehat{G} \longrightarrow G$ be the inverse isomorphism. Let $i \in I \setminus J$. Then $\pi_i \xi : \widehat{G} \longrightarrow G \longrightarrow P$ is an epimorphism. By 3.14 its kernel is C_j for some $j \in J$. Put $\alpha(i) = j$.

Since both $\pi_i \xi$ and π_j are epimorphisms $\widehat{G} \longrightarrow P$ whose kernel is C_j, it follows that $\pi_i \xi = \beta_i \pi_j$ for appropriate $\beta_i \in \operatorname{Aut}(P)$. Now let $\bar{g} = (g_i \mid i \in I) \in G$, then $\bar{g} = \xi(\widehat{g})$ for some $\widehat{g} \in \widehat{G}$. Hence $g_i = \pi_i(\bar{g}) = \pi_i \xi(\widehat{g}) = \beta_i \pi_j(\widehat{g}) = \beta_i(g_j) = \beta_i(g_{\alpha(i)})$, for every $i \in I \setminus J$. As $\widehat{g} = \pi_J(\bar{g}) \in \widehat{G} = P(J, V)$ and $g_i = \beta_i(g_{\alpha(i)})$ for all $i \in I \setminus J$, we see that $\bar{g} \in P(I, J, V, \alpha, \beta)$. So $G \subseteq P(I, J, V, \alpha, \beta)$.

Conversely, let $\bar{g} \in P(I, J, V, \alpha, \beta)$. Then $\widehat{g} = \pi_J(\bar{g}) \in \widehat{G} = P(J, V) = \pi_J(G)$, so there exists $h = (h_i \mid i \in I) \in G$ such that $\pi_J(h) = \widehat{g}$, whence $h_j = g_j$ for all $j \in J$. By the preceding paragraph, $h_i = \beta_i(h_{\alpha(i)}) = \beta_i(g_{\alpha(i)})$ for all $i \in I \setminus J$. On the other hand, for all $i \in I \setminus J$ we have $g_i = \beta_i(g_{\alpha(i)})$, because $\bar{g} \in P(I, J, V, \alpha, \beta)$. Thus $h_i = g_i$ for all $i \in I \setminus J$, whence $h = \bar{g}$, and finally $P(I, J, V, \alpha, \beta) \subseteq G$. $\square$

Subdirect products of several A

Lemma 3.17. *Let*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

be a commutative diagram of groups such that β and γ are epimorphisms. Let $C_1 \subseteq C$ be a subgroup, $A_1 = \gamma^{-1}(C_1)$, $D_1 = \delta(C_1)$, and $B_1 = \beta^{-1}(D_1)$. Then:

1) $B_1 = \alpha(A_1) \operatorname{Ker} \beta$.
2) *If* $\alpha(\operatorname{Ker} \gamma) = \operatorname{Ker} \beta$, *then* $\alpha(A_1) = B_1$.

(Of course, A of this lemma is not the group A of Theorem 1.1.)

Here and below we denote by $\mathfrak{E}_p$ the class of all elementary abelian p-groups, so that $\mathfrak{E}_2 = \mathfrak{E}$.

Lemma 3.18.

1) *Let D be a group and $B, C \trianglelefteq D$. Suppose that $BC = D$, $[B, C] = 1$, and $B \in \mathfrak{E}_p$. Then there exists a $B_1 \subseteq B$ such that $D = B_1 \times C$.*
2) *Let B be an elementary abelian p-group, C be an arbitrary group, and $D = B \times C$. Suppose $H \trianglelefteq D$ and $H \in \mathfrak{E}_p$. Then $D/H \cong E \times (C/F)$, where $E, F \in \mathfrak{E}_p$.*

Proof. 1) Since $BC = D$, $[B, C] = 1$ and B is abelian, it follows that B commutes with every element of D, i.e., $B \subseteq Z(D)$. As $B \in \mathfrak{E}_p$, any subgroup of B has a complement in B; in particular, there exists $B_1 \subseteq B$ such that $B = B_1 \times (B \cap C)$. Now $B_1 C = B_1((B \cap C)C) = (B_1(B \cap C))C = BC = D$; $B_1 \trianglelefteq D$, because $B_1 \subseteq Z(D)$; $[B_1, C] \subseteq [B, C] = 1$; and $B_1 \cap C = (B_1 \cap B) \cap C = B_1 \cap (B \cap C) = 1$. Thus, $D = B_1 \times C$.

2) Let $\varphi : D \longrightarrow D/H$ be the canonical epimorphism. Obviously, $D/H = \varphi(B)\varphi(C)$, $[\varphi(B), \varphi(C)] = \varphi([B, C]) = 1$, and $\varphi(B) \in \mathfrak{E}_p$. By 1), there exists $E \subseteq \varphi(B)$ such that $D/H = E \times \varphi(C)$. Next, $\varphi(C) \cong C/C \cap H = C/F$, where $F = C \cap H$. It remains to note that $E, F \in \mathfrak{E}_p$. $\square$

3.19. 1) *If $V \subseteq k^J$ is a wide subspace, then $Q(J, V) \subseteq Q^J$ is a reduced wide subgroup (and is, therefore, a subdirect product of several copies of A).*

2) *Any subdirect product of several copies of A is isomorphic to a group of the form $Q(J, V) \times U$, where $V \subseteq k^J$ is a wide subspace and $U \in \mathfrak{E}$ is an elementary abelian 2-group.*

Proof. 1) It follows from Corollary 3.4 that $Q(J,V)$ is wide. Next, for any $i \in J$ and $g \in L_i$, $g \neq 1$, the element $\rho_i(g)$ is nontrivial and is in $\operatorname{Ker}\pi_{J'}$, where $J' = J \setminus \{i\}$. So $Q(J,V)$ is reduced.

2) Let G be such a product. We can assume that $G \subseteq Q^I$ is a reduced wide subgroup, for some I. As Q is monolithic, we can apply 3.6 with $G_i = Q_i$, $\overline{G}_i = P_i$, and $\varphi_i = \tau_i$. We see that $G = \tau_I^{-1}(H)$, where $H \subseteq P^I$ is a wide subgroup. By 3.16, we have $H = P(I, J, V, \alpha, \beta)$ for some J, V, α, β.

Since any automorphism $f \in \operatorname{Aut}(P)$ is inner, it can be lifted with respect to τ to an automorphism $\widehat{f} \in \operatorname{Aut}(Q)$ (i.e., $\tau(\widehat{f}(g)) = f(\tau(g))$ for every $g \in Q$). For each $i \in I \setminus J$ choose a lifting $\widehat{\beta}_i$ for β_i.

Consider $\mathcal{P}$-objects $X_1 = (J, \{P_j \mid j \in J\})$, $X_2 = (I, \{P_i \mid i \in I\})$, $Y_1 = (J, \{Q_j \mid j \in J\})$, and $Y_2 = (I, \{Q_i \mid i \in I\})$, and the diagram

$$\begin{tikzcd} Y_1 \arrow[r, "\zeta"] \arrow[d, "\gamma"'] & Y_2 \arrow[d, "\delta"] \\ X_1 \arrow[r, "\xi"] & X_2 \end{tikzcd}$$

where $\gamma = (\mathrm{id}_J, (\tau_j \mid j \in J))$, $\delta = (\mathrm{id}_I, (\tau_i \mid i \in I))$, $\xi = (a, \theta)$, where $a : I \longrightarrow J$ equals id_J on J and equals α on $I \setminus J$; $\theta = (\theta_i \mid i \in I)$, $\theta_j = \mathrm{id}_P$ for $j \in J$, $\theta_i = \beta_i$ for $i \in I \setminus J$, and $\zeta = (a, \widehat{\theta})$, where $\theta_j = \mathrm{id}_Q$ when $j \in J$ and $\theta_i = \widehat{\beta}_i$ when $i \in I \setminus J$. It is easy to see that this diagram is commutative. By applying the functor $\prod$ we obtain the commutative diagram of groups

$$\begin{tikzcd} Q^J \arrow[r, "\zeta_*"] \arrow[d, "\tau_J"'] & Q^I \arrow[d, "\tau_I"] \\ P^J \arrow[r, "\xi_*"] & P^I. \end{tikzcd}$$

Since $\widehat{\sigma} = \sigma\tau$, it follows that $\widehat{\sigma}_I = \sigma_I \tau_I$. Hence $\tau_J^{-1}(P(J,V)) = \tau_J^{-1}(\sigma_J^{-1}(V)) = (\sigma_J \tau_J)^{-1}(V) = \widehat{\sigma}_J^{-1}(V) = Q(J,V)$.

Next, $\xi_*(P(J,V))$ is nothing else but $P(I,J,V,\alpha,\beta) = H$. Next, note that $\operatorname{Ker}\tau_I = \widehat{K}^I$. By applying Lemma 3.17, 1) to the latter diagram with $A = Q^J, \dots, C_1 = P(J,V)$, we find that $G = \tau_I^{-1}(H) = R\widehat{K}^I$, where $R = \zeta_* \tau_J^{-1}(C_1) = \zeta_*(Q(J,V))$. It is easy to see that $\pi_J \zeta_* = \mathrm{id}_{Q^J}$, so ζ_* is an embedding and $R \cong Q(J,V)$. Further, $\widehat{K}^I$ is an elementary abelian 2-group contained in the center of Q^I whence in the center of G. It follows from Lemma 3.18 that $G = R \times B$, where $B \subseteq \widehat{K}^I$ is an elementary abelian 2-group. $\square$

Groups of $\mathfrak{P}$ *and* $\mathfrak{Q}$ In this paragraph we describe the groups of formations $\mathfrak{P} = \operatorname{form}(S_5)$ and $\mathfrak{Q} = \operatorname{form}(A)$ in an explicit form.

For $i \in I$ denote by e_i the usual basis vectors of k^I, i.e., $e_i = (v_j \mid j \in I)$, where $v_i = 1$ and $v_j = 0$ for $j \neq i$. Moreover, by $\delta_i : k^I \longrightarrow k^{I \setminus \{i\}}$ we denote the usual projection (the removal of i-th coordinate); i.e., $\delta_i = \pi_{I \setminus \{i\}}$.

3.20. 1) *Suppose G_i, $i \in I$, are nonabelian simple groups and $G_I = \prod_{i \in I} G_i$. Then every normal subgroup of G_I is $\widehat{G}_J = \rho_J(G_J)$ for appropriate $J \subseteq I$.*

2) *A minimal normal subgroup of $P(I,V)$ is $\widehat{N}_i$ for some $i \in I$.*

Proof. 1) This is clear.

2) Obviously, each $\widehat{N}_i$ is a minimal normal subgroup of $P(I,V)$. Prove the converse.

Suppose $X \trianglelefteq P(I,V)$ is minimal. Then either $X \cap \widehat{N}^I = 1$ or $X \subseteq \widehat{N}^I$. Assume $X \cap \widehat{N}^I = 1$. Then X is isomorphic to a subgroup of $P(I,V)/\widehat{N}^I \cong V$ and so is abelian. So $\pi_i(X)$ is an abelian normal subgroup of $P_i \cong S_5$ for each $i \in I$, whence $\pi_i(X) = 1$. Since the latter is true for all $i \in I$, it follows that $X = 1$, a contradiction. Thus, $X \subseteq \widehat{N}^I$. By 1), any normal subgroup of $\widehat{N}^I$ is $\widehat{N}^J$ for appropriate $J \subseteq I$ and so contains $\widehat{N}_i$ for some $i \in I$. So $X = \widehat{N}_i$, as required. $\square$

3.21. *$P(I, V)/\widehat{N}_i \cong P(I \setminus \{i\}, \delta_i(V)) \times T$, where $T \cong Z_2$ when $e_i \in V$ and $T = 1$ otherwise.*

Proof. Denote $J = I \setminus \{i\}$. We have commutative diagram

$$\begin{tikzcd} P^I \arrow[r, "\pi_J"] \arrow[d, "\sigma_I"'] & P^J \arrow[d, "\sigma_J"] \\ k^I \arrow[r, "\delta_i"] & k^J. \end{tikzcd}$$

Note that $\pi_J(\widehat{N}^I) = \widehat{N}^J$ and apply Lemma 3.17, 2) to this diagram and $C_1 = V \subseteq k^I$. We see that $\pi_J(P(I, V)) = P(J, \delta_i(V))$.

Note that $\operatorname{Ker}\pi_J = \widehat{P}_i$, so $P(I, V) \cap \operatorname{Ker}\pi_J = P(I, V) \cap \widehat{P}_i$. Obviously, $P(I, V) \cap \widehat{P}_i = \widehat{P}_i$ when $e_i \in V$ and $= \widehat{N}_i$ when $e_i \notin V$. It follows that $P(I, V)/\widehat{N}_i \cong \pi_J(P(I, V)) = P(J, \delta_i(V))$ when $e_i \notin V$. On the other hand, if $e_i \in V$, then $\widehat{P}_i \subseteq P(I, V)$ and $P(I, V) = \pi_J(P(I, V)) \times \widehat{P}_i = P(J, \delta_i(V)) \times \widehat{P}_i$, whence $P(I, V)/\widehat{N}_i = P(J, \delta_i(V)) \times (\widehat{P}_i/\widehat{N}_i) \cong P(J, \delta_i(V)) \times Z_2$. $\square$

Lemma 3.22. *Let $G \subseteq G_I = \prod_{i\in I} G_i$ be a wide subgroup, and $D \trianglelefteq G$ be its normal subgroup. Let $i \in I$, and $H_i \trianglelefteq G_i$ be a normal subgroup. Suppose that $[H_i, H_i] = H_i$, $\pi_i(D) \supseteq H_i$, and $\widehat{H}_i = \rho_i(H_i) \subseteq G$. Then $\widehat{H}_i \subseteq D$.*

Proof. It is sufficient to prove that $[\widehat{H}_i, D] = \widehat{H}_i$, because then $\widehat{H}_i = [\widehat{H}_i, D] \subseteq D$.

Take $x, y \in H_i$. As $\pi_i(D) \supseteq H_i$, there exists $z \in D$ such that $\pi_i(z) = y$. Now $\rho_i([x, y]) = \rho_i(x)\rho_i(yxy^{-1}) = \rho_i(x)\rho_i(\pi_i(z)x\pi_i(z)^{-1}) = \rho_i(x)(z\rho_i(x)z^{-1}) = [\rho_i(x), z] \in [\widehat{H}_i, D]$. Hence $\widehat{H}_i = \rho_i(H_i) = \rho_i([H_i, H_i]) \subseteq [\widehat{H}_i, D]$. The inverse inclusion $[\widehat{H}_i, D] \subseteq \widehat{H}_i$ is clear, because $\widehat{H}_i$ is normal in G_I. $\square$

Proposition 3.23. *The groups of $\mathfrak{P}$ are precisely the groups isomorphic to the groups of the form $P(I, V) \times U$, where $V \subseteq k^I$ is wide and $U \in \mathfrak{E}$.*

Proof. It follows from the statements 3.10 and 3.13 that the groups of $\mathfrak{P}$ are precisely the groups having a representation of the form $G \cong P(I, V)/H$. Therefore these are precisely the groups with a representation of the form $G \cong P(I, V) \times U/H$. For a given G take a representation of this form with minimal possible $|I|$ and minimal $|U|$ when $|I|$ is fixed. It remains to show that $H = 1$.

We consider $P^I \times U$ as a direct product of P_i, $i \in I$, and U. Thus $G = P(I, V) \times U$ is a wide subgroup in $P^I \times U$. Note that $\widehat{N}_i \subseteq G$ for all $i \in I$. Further, $\pi_i(H) \trianglelefteq P_i$. If $\pi_i(H) \neq 1$, then $\pi_i(H) \supseteq N_i$. Since $[N_i, N_i] = N_i$ and $\widehat{N}_i \subseteq G$, Lemma 3.22 applies, whence $\widehat{N}_i \subseteq H$. So G is a quotient group of

$$P(I, V) \times U/\widehat{N}_i \cong \bigl(P(I, V)/\widehat{N}_i\bigr) \times U \cong P\bigl(I \setminus \{i\}, \delta_i(V)\bigr) \times T \times U,$$

where $T = 1$ or Z_2. But this contradicts the minimality of $|I|$. Thus always $\pi_i(H) = 1$, whence $H \subseteq U$. But then $G \cong P(I, V) \times (U/H)$, which contradicts the minimality of U if $H \neq 1$. $\square$

Corollary 3.24. *$\emptyset$, $\mathfrak{I}$, $\mathfrak{E}$, and $\mathfrak{P}$ are precisely all distinct subformations of $\mathfrak{P}$.*

Proof. Let $\mathfrak{F}$ be a subformation of $\mathfrak{P}$. If $\mathfrak{F} \subseteq \mathfrak{E}$, then $\mathfrak{F} = \emptyset, \mathfrak{I}$, or $\mathfrak{E}$. Otherwise $\mathfrak{F}$ contains a group of the form $P(I, V)$. Since S_5 is a quotient of $P(I, V)$, we obtain $\mathfrak{F} \ni S_5$, so $\mathfrak{F} = \mathfrak{P}$. $\square$

3.25. *$Q(I, V)/\widehat{L}_i \cong Q(I \setminus \{i\}, \delta_i(V)) \times T$, where $T = 1$ or Z_2 for $e_i \notin V$ or $e_i \in V$, respectively.*

Proof. The proof is precisely similar to that of 3.21. The details are left to the reader. □

Let I and V be as earlier, and $W \subseteq k^I$ be a subspace. Define $R(I,V,W) = Q(I,V)/\nu_I(W)$. To abbreviate the notation we often write $K(W)$ for $\nu_I(W)$.

3.26. *Let $Y = Q(I,V)$. Then*:

1) $\widehat{L}^I = Y^{\mathfrak{E}} = [Y,Y] = O^2(Y)$.
2) $\widehat{K}^I = Y^{\mathfrak{P}} = Z(Y) = O_2(Y)$.

Proof. 1) Since $Y/\widehat{L}^I \cong V \in \mathfrak{E}$, it follows that $Y^{\mathfrak{E}} \subseteq \widehat{L}^I$. Next, $Y^{\mathfrak{E}}$ is the intersection of kernels of all epimorphisms $\theta : Y \longrightarrow Z_2$. As there exist no nontrivial homomorphisms from $\widehat{L}_i \cong SL(2,5)$ to Z_2, we see that $\widehat{L}_i \subseteq \operatorname{Ker}\theta$ for all i and θ, whence $\widehat{L}^I \subseteq \operatorname{Ker}\theta$ for any θ, whence $\widehat{L}^I \subseteq Y^{\mathfrak{E}}$. Thus, $Y^{\mathfrak{E}} = \widehat{L}^I$.

Since $Y/\widehat{L}^I \in \mathfrak{E}$, it follows that $\widehat{L}^I \supseteq [Y,Y], O^2(Y)$. On the other hand, $[\widehat{L}_i, \widehat{L}_i] = \widehat{L}_i$, so $[\widehat{L}^I, \widehat{L}^I] = \widehat{L}^I$, whence $\widehat{L}^I \subseteq [Y,Y], O^2(Y)$. So $\widehat{L}^I$ coincides with both $[Y,Y]$ and $O^2(Y)$.

2) $Y/\widehat{K}^I \cong P(I,V) \in \mathfrak{P}$, so $Y^{\mathfrak{P}} \subseteq \widehat{K}^I$. Any group of $\mathfrak{P}$ is of the form $P(J,U)$ or $P(J,U) \times W$ and so is a subdirect product of several groups isomorphic to either S_5 or Z_2. So $Y^{\mathfrak{P}}$ equals the intersection of kernels of all epimorphisms $\theta : Y \longrightarrow X$, where $X = Z_2$ or S_5. Observe that $Z(\widehat{L}_i) = \widehat{K}_i$ is contained in the kernel of every homomorphism from $\widehat{L}_i \cong SL(2,5)$ to either Z_2 or S_5. So $\widehat{K}_i \subseteq \operatorname{Ker}\theta$ for any i and θ. So $\widehat{K}^I \subseteq \operatorname{Ker}\theta$ for each θ whence $\widehat{K}^I \subseteq Y^{\mathfrak{P}}$. Hence $Y^{\mathfrak{P}} = \widehat{K}^I$.

Further, $\widehat{K}^I$ is an elementary abelian 2-group that lies in the center of Q^I and so in the center of $Q(I,V)$. Hence $\widehat{K}^I \subseteq Z(Y), O_2(Y)$. On the other hand, let $X = Z(Y)$ or $O_2(Y)$. Then $\pi_i(X) \subseteq Z(Q_i)$ or $\subseteq O_2(Q_i)$ respectively, whence $\pi_i(X) \subseteq K_i$. Since $\pi_i(X) \subseteq K_i$ for all i, it follows that $X \subseteq \widehat{K}^I$. So $X = \widehat{K}^I$. □

3.27. *$\mathfrak{Q}$ is the class of all groups that are isomorphic to the groups $R(I,V,W) \times U$, where $V \subseteq k^I$ is wide and $U \in \mathfrak{E}$.*

Proof. Any group of this form lies in $\mathfrak{Q}$. Conversely, suppose $G \in \mathfrak{Q}$. It follows from 3.19 and 3.10 that G has a representation of the form $G \cong Q(I,V) \times U/H$, where V is wide. Choose a representation with minimal $|I|$ and, moreover, with minimal $|U|$ when $|I|$ is fixed. Now it is sufficient to show that $H = K(W) \times \{1\}$ for some $W \subseteq k^I$.

Similarly to the proof of Proposition 3.23, we consider $G = Q(I,V) \times U$ as a wide subgroup in $Q^I \times U$. We have $\widehat{L}_i \subseteq G$ for all $i \in I$. Further, $\pi_i(H) \trianglelefteq Q_i$. If $\pi_i(H) \nsubseteq K_i$, then $\pi_i(H) \supseteq L_i$. Since $[L_i, L_i] = L_i$ and $\widehat{L}_i \subseteq G$, Lemma 3.22 applies, whence $\widehat{L}_i \subseteq H$. Then G is a quotient for $(Q(I,V) \times U)/\widehat{L}_i \cong (Q(I,V)/\widehat{L}_i) \times U \cong (Q(I \setminus \{i\}, \delta_i(V)) \times T) \times U \cong Q(I \setminus \{i\}, \delta_i(V)) \times (T \times U)$, where $T = 1$ or Z_2. This contradicts the minimality of $|I|$. Thus $\pi_i(H) \subseteq K_i$.

Since the projection of H to each Q_i is 1 or Z_2, and $U \in \mathfrak{E}$, it follows that $H \in \mathfrak{E}$. By applying Lemma 3.18 we obtain $G \cong Q(I,V)/W_1 \times U_1$, where both $W_1, U_1 \in \mathfrak{E}$. Next, $W_1 \subseteq O_2(Q(I,V)) = \widehat{K}^I$ by 3.26. But the subgroups of $\widehat{K}^I$ are precisely all subgroups of the form $K(W)$, where $W \subseteq k^I$ is a subspace. □

3.28. *Any subformation of $\mathfrak{Q}$ is either contained in $\mathfrak{E}$ or contains $\mathfrak{P}$.*

Proof. Let $\mathfrak{F} \subseteq \mathfrak{Q}$ be a subformation such that $\mathfrak{F} \nsubseteq \mathfrak{E}$. Then $\mathfrak{F}$ contains a group of the form $R(I,V,W) \times U$ with $I \neq \emptyset$ and so contains $R(I,V,W)$. Since $P(I,V)$ is a quotient of $R(I,V,W)$ and S_5 is a quotient of $P(I,V)$, it follows that $S_5 \in \mathfrak{F}$, whence $\mathfrak{F} \supseteq \mathfrak{P}$. □

3.29. *Let $R = R(I,V,W)$. Then $\widehat{L}^I/K(W) = R^{\mathfrak{E}} = [R,R] = O^2(R)$ and $\widehat{K}^I/K(W) = R^{\mathfrak{P}} = O_2(R) \subseteq Z(R)$.*

Proof. First note that if X is an arbitrary group, $S(X)$ means one of $[X,X]$, $O^2(X)$, or $O_2(X)$, and $Y \trianglelefteq X$ is a normal subgroup such that $Y \subseteq S(X)$, then $S(X/Y) = S(X)/Y$. Also, if $Y \subseteq Z(X)$, then $Z(X/Y) \supseteq Z(X)/Y$. Finally, it follows from 3.12 that if $\mathfrak{F}$ is a formation such that $X^{\mathfrak{F}} \supseteq Y$, then $(X/Y)^{\mathfrak{F}} = X^{\mathfrak{F}}/Y$.

Now it is sufficient to apply these observations for $X = Q(I,V)$, $Y = K(W)$, $\mathfrak{F} = \mathfrak{E}, \mathfrak{P}$, and to use 3.26. □

For $R = R(I,V,W)$ we denote $T(R) = \widehat{K}^I/K(W)$.

Epimorphisms of groups $R(I,V,W)$ Though it is possible to give a complete description of epimorphisms between groups $R(I,V,W)$ we prove only statements that are strictly necessary for the proof of Theorem 2.4.

3.30. *Suppose that $V \subseteq k^I$ and $U \subseteq k^J$ are wide and $\alpha : J \longrightarrow I$ is an embedding such that $\alpha^*(V) = U$. Then there exists an epimorphism $\psi = \psi(\alpha) : Q(I,V) \longrightarrow Q(J,U)$ such that the diagram*

$$\begin{array}{ccc} k^I & \xrightarrow{\alpha^*} & k^J \\ \Big\downarrow{\scriptstyle \nu_I} & & \Big\downarrow{\scriptstyle \nu_J} \\ Q(I,V) & \xrightarrow{\psi} & Q(J,U) \end{array} \tag{1}$$

commutes.

Proof. Both squares of the diagram

$$\begin{array}{ccccc} k^I & \xrightarrow{\nu_I} & Q^I & \xrightarrow{\widehat{\sigma}_I} & k^I \\ \Big\downarrow{\scriptstyle \alpha_k^*} & & \Big\downarrow{\scriptstyle \alpha_Q^*} & & \Big\downarrow{\scriptstyle \alpha_k^*} \\ k^J & \xrightarrow{\nu_J} & Q^J & \xrightarrow{\widehat{\sigma}_J} & k^J \end{array} \tag{2}$$

commute, by 3.1. Next, $\nu_I(k_I) = \widehat{K}^I \subseteq Q(I,V)$, and similarly $\nu_J(k_J) \subseteq Q(J,V)$. So it suffices to prove that $\alpha_Q^*(Q(I,V)) = Q(J,V)$; then we may take $\psi = \alpha_Q^*|_{Q(I,V)}$. To do this it is sufficient to apply Lemma 3.17 to the right square (or, rather, to its transpose) with $A = Q^I$, $B = Q^J$, $C = k^I$, $D = k^J$, and $C_1 = V$. □

Lemma 3.31. *Let X_i, $i \in I$ and Y_j, $j \in J$ be nonabelian simple groups, $X_I = \prod_{i\in I} X_i$, $Y_J = \prod_{j\in J} Y_j$, and $\varphi : X_I \longrightarrow Y_J$ be an epimorphism. Then there exists an injection $f : J \longrightarrow I$ such that $\varphi(\widehat{X}_i) = \widehat{Y}_j$ when $i = f(j)$ and $\varphi(\widehat{X}_i) = 1$ when $i \notin f(J)$.*

Proof. Since $\widehat{X}_i \trianglelefteq X_I$ and since φ is an epimorphism, it follows that $\varphi(\widehat{X}_i) \trianglelefteq Y_J$. As $\widehat{X}_i$ is simple, we see that either $\varphi(\widehat{X}_i) = 1$ or $\varphi(\widehat{X}_i)$ is a minimal normal subgroup of Y_J, whence $\varphi(\widehat{X}_i) = \widehat{Y}_j$ for some j. Since φ is an epimorphism, it follows that the subgroups $\varphi(\widehat{X}_i)$ generate Y_J, so for every $j \in J$ there exists an $i \in I$ such that $\varphi(\widehat{X}_i) = \widehat{Y}_j$. This i is unique because $\varphi(\widehat{X}_i)$ and $\varphi(\widehat{X}_l)$ must commute when $i \neq l$, but $\widehat{Y}_j$ is nonabelian. It remains to put $f(j) = i$. □

3.32. *Let $R_1 = R(I_1,V_1,W_1)$, $R_2 = R(I_2,V_2,W_2)$, and suppose $\psi : R_1 \longrightarrow R_2$ is an epimorphism. Then there exists an injection $\alpha : I_2 \longrightarrow I_1$ such that $\alpha^*(V_1) = V_2$, $\alpha^*(W_1) \subseteq W_2$, and $T(R_1) \cap \operatorname{Ker}\psi = K(W')/K(W)$, where $W' = (\alpha^*)^{-1}(W_2)$.*

Proof. We divide the proof into several steps.

1) Denote $K^l = \widehat{K}^{I_l}/K(W_l)$ $(= T(R_l))$ and $L^l = \widehat{L}^{I_l}/K(W_l)$, $l = 1,2$. We use superscripts to avoid confusion with the notation for the subgroups K_i, $i \in I$; and we use some similar notation below. We have $R_l^{\mathfrak{E}} = L^l$ and $R_l^{\mathfrak{P}} = K^l$ by 3.29. It follows from 3.12 that $\psi(L^1) = L^2$ and $\psi(K^1) = K^2$.

2) Denote $P^l = P(I_l, V_l)$. Let $\zeta_l : L^l \longrightarrow R_l$ be the embedding of subgroups, and $\psi' = \psi|_{L^1}$. Also we define epimorphisms $\tau^l : R_l \longrightarrow P^l$ by $\tau^l(gK(W_l)) = \tau_{I_l}(g)$, where $g \in Q(I_l, V_l)$. Then $\operatorname{Ker}\tau^l = K^l$. As $\psi(K^1) = K^2$, there exists an epimorphism $\overline{\psi} : P^1 \longrightarrow P^2$ such that the diagram

$$\begin{tikzcd}
L^1 \arrow[r, "\zeta_1"] \arrow[d, "\psi'"'] & R_1 \arrow[r, "\tau^1"] \arrow[d, "\psi"'] & P^1 \arrow[d, "\overline{\psi}"] \\
L^2 \arrow[r, "\zeta_2"] & R_2 \arrow[r, "\tau^2"] & P^2
\end{tikzcd} \tag{3}$$

is commutative.

3) For every $i \in I_1$ let $\bar{L}_i = \widehat{L}_i K(W_1)/K(W_1)$ be the image of $\widehat{L}_i \subseteq Q(I_1, V_1)$ in R_1. Define $\bar{L}_i$ for $i \in I_2$ in the same way. Since $\widehat{L}_i \trianglelefteq Q(I_l, V_l)$, it follows that $\bar{L}_i \trianglelefteq R_l$, $i \in I_l$.

Further, $\tau^l(\bar{L}_i) = \tau^l(\widehat{L}_i K(W_1)/K(W_1)) = \tau_{I_l}(\widehat{L}_i)$ by the definition of τ^l. The latter group is $\widehat{N}_i \subseteq P^l$. Thus, $\tau^l(\bar{L}_i) = \widehat{N}_i$. In particular, τ^l takes $L^l = \langle \bar{L}_i \mid i \in I_l\rangle$ to $\langle \widehat{N}_i \mid i \in I_l\rangle = \widehat{N}^{I_l}$. (The latter fact can be also proved in a different way if we observe that $\widehat{N}^{I_l} = (P^l)^{\mathfrak{E}}$ and apply 3.12 to $\tau^l : R_l \longrightarrow P^l$.)

4) Denote $N^l = \widehat{N}^{I_l}$. It follows from 3.12 and $N^l = (P^l)^{\mathfrak{E}}$ that $\overline{\psi}(N^1) = N^2$. As $\widehat{N}_i \cong A_5$ for all i, Lemma 3.31 implies that there exists an embedding $\alpha : I_2 \longrightarrow I_1$ such that $\overline{\psi}(\widehat{N}_{\alpha(j)}) = \widehat{N}_j$ when $j \in I_2$ and $\overline{\psi}(\widehat{N}_i) = 1$ when $i \in I_1 \setminus \alpha(I_2)$. It remains to show that this α satisfies the conclusion of the statement.

5) Show that $\psi'(\bar{L}_{\alpha(j)}) = \bar{L}_j$ when $j \in I_2$ and $\psi'(\bar{L}_i) = 1$ when $i \in I_1 \setminus \alpha(I_2)$.

For $i \in I_l$ let $M_i = (\tau^l)^{-1}(\widehat{N}_i)$ be the preimage of $\widehat{N}_i \subseteq P^l$ in R_l with respect to τ^l. Since $\tau^l(\bar{L}_i) = \widehat{N}_i$ and $\operatorname{Ker}\tau^l = K^l$ we have $M_i = K^l \bar{L}_i$.

Note that $\psi(M_i) \supseteq \psi(K^1) = K^2 = \operatorname{Ker}\tau^2$ for any $i \in I_1$. Therefore $\psi(M_i) = (\tau^2)^{-1}(\tau^2\psi(M_i)) = (\tau^2)^{-1}(\overline{\psi}\tau^1(M_i)) = (\tau^2)^{-1}(\overline{\psi}(\widehat{N}_i))$. So $\psi(M_i) = (\tau^2)^{-1}(1) = K^2$ when $i \in I_1 \setminus \alpha(I_2)$ and $\psi(M_i) = (\tau^2)^{-1}(\widehat{N}_j) = M_j$ when $i = \alpha(j)$.

Next, $K^l \subseteq Z(R_l)$ and so $K^l \subseteq Z(M_i)$. Moreover, $\bar{L}_i$ is a homomorphic image of $\widehat{L}_i \cong SL(2,5)$ whence $[\bar{L}_i, \bar{L}_i] = \bar{L}_i$. So $[M_i, M_i] = [K^l\bar{L}_i, K^l\bar{L}_i] = [\bar{L}_i, \bar{L}_i] = \bar{L}_i$. Now, if $i \in I_1 \setminus \alpha(I_2)$, then

$$\psi'(\bar{L}_i) = \psi\big([M_i, M_i]\big) = \big[\psi(M_i), \psi(M_i)\big] = \big[K^2, K^2\big] = 1.$$

If $i = \alpha(j)$, then

$$\psi'(\bar{L}_i) = \big[\psi(M_i), \psi(M_i)\big] = [M_j, M_j] = \bar{L}_j,$$

as required.

6) Consider the map $\mu_l : k^{I_l} \longrightarrow R_l$ defined by $\mu_l(v) = \nu_{I_l}(v)K(W_l)$. We prove that the diagram

$$\begin{tikzcd}
k^{I_1} \arrow[r, "\mu_1"] \arrow[d, "\alpha^*"'] & R_1 \arrow[d, "\psi"] \\
k^{I_2} \arrow[r, "\mu_2"] & R_2
\end{tikzcd} \tag{4}$$

is commutative. It is sufficient to check that

$$\mu_2\big(\alpha^*(e_i)\big) = \psi\big(\mu_1(e_i)\big) \tag{5}$$

for every $i \in I_1$.

Let t_i be the unique nontrivial central element of $\widehat{L}_i$, $i \in I_l$, and let $\bar{t}_i = t_i K(W_l)$ be its image in $\bar{L}_i$ (may be $\bar{t}_i = 1$). Obviously $\nu_{I_l}(e_i) = t_i$, $i \in I_l$, so $\mu_l(e_i) = \bar{t}_i$.

Since $\bar{L}_i$ is a quotient of $\widehat{L}_i \cong SL(2,5)$ and has $\widehat{N}_i \cong A_5$ as a quotient, it follows that $\bar{L}_i \cong A_5$ or $SL(2,5)$. Since t_i is central in $\widehat{L}_i$, we see that $\bar{t}_i$ is central in $\bar{L}_i$. So $\bar{t}_i = 1$ when $\bar{L}_i \cong A_5$. On the other hand, if $\bar{L}_i \cong SL(2,5)$, then $\widehat{L}_i$ is mapped onto $\bar{L}_i$ isomorphically and the only nontrivial central element of $\widehat{L}_i$ goes to the element of $\bar{L}_i$ with the same properties. That is, if $\bar{L}_i \cong SL(2,5)$, then $\bar{t}_i$ is the only nontrivial central element of $\bar{L}_i$.

Now we prove (5). First suppose $i \in I_1 \setminus \alpha(I_2)$. Then $\alpha^*(e_i) = 0$ and $\mu_2(\alpha^*(e_i)) = 1$. Also $\psi(\bar{L}_i) = 1$, whence $\psi(\mu_1(e_i)) = \psi(\bar{t}_i) = 1$. I.e., both sides of (5) are trivial.

Now let $i = \alpha(j)$, $j \in I_2$. Then $\alpha^*(e_i) = e_j$ and $\mu_2(\alpha^*(e_i)) = \bar{t}_j$. So we need to prove that $\psi(\bar{t}_i) = \bar{t}_j$. As both $\bar{L}_i$ and $\bar{L}_j$ are isomorphic to either A_5 or $SL(2,5)$ and since $\bar{L}_j$ is a homomorphic image of $\bar{L}_i$, we have three possibilities: (a) $\bar{L}_i \cong \bar{L}_j \cong A_5$, (b) $\bar{L}_i \cong SL(2,5)$, $\bar{L}_j \cong A_5$, and (c) $\bar{L}_i \cong \bar{L}_j \cong SL(2,5)$. In the case (a) we have $\bar{t}_i = 1$ and $\bar{t}_j = 1$ whence $\psi(\bar{t}_i) = \bar{t}_j$. In the case (b) we have $\bar{t}_j = 1$; also $\psi(\bar{t}_i)$ must be a central element of $\psi(\bar{L}_i) = \bar{L}_j$, whence $\psi(\bar{t}_i) = 1 = \bar{t}_j$. Finally, in the case (c) ψ maps $\bar{L}_i$ onto $\bar{L}_j$ isomorphically and so the nontrivial central element of $\bar{L}_i$ goes to a nontrivial central element of $\bar{L}_j$, i.e., $\bar{t}_i$ goes to $\bar{t}_j$.

7) Obviously, $W_l = \operatorname{Ker}\mu_l$. Since the diagram (4) is commutative, it follows that $\alpha^*(\operatorname{Ker}\mu_1) \subseteq \operatorname{Ker}\mu_2$, i.e., $\alpha^*(W_1) \subseteq W_2$. Next, let $S = T(R_1) \cap \operatorname{Ker}\psi$. Since $T(R_1) = \operatorname{Im}\mu_1$, it follows that $S = \mu_1(\operatorname{Ker}\psi\mu_1) = \mu_1(\operatorname{Ker}\mu_2\alpha^*) = \mu_1((\alpha^*)^{-1}(\operatorname{Ker}\mu_2)) = \mu_1((\alpha^*)^{-1}(W_2)) = \mu_1(W') = K(W')/K(W)$. Thus we have proved two of the three properties of α.

8) For $i \in I_l$ let $\widehat{\pi}_i$ be the restriction of π_i to P^l. Let $C_i = \operatorname{Ker}\widehat{\pi}_i = P^l \cap \operatorname{Ker}\pi_i$ as in 3.14. We show that $\overline{\psi}^{-1}(C_j) = C_{\alpha(j)}$ for any $j \in I_2$.

The composite map $P^1 \xrightarrow{\overline{\psi}} P^2 \xrightarrow{\widehat{\pi}_j} S_5$ is an epimorphism of P^1 onto S_5. So its kernel equals C_i for some $i \in I_1$ by 3.14. On the other hand, this kernel equals $\overline{\psi}^{-1}(\operatorname{Ker}\widehat{\pi}_j) = \overline{\psi}^{-1}(C_j)$. It remains to prove that $i = \alpha(j)$. Note that $\widehat{N}_i$ is the only of all subgroups of the form $\widehat{N}_h$, $h \in I_1$, that is not contained in C_i. On the other hand, $\widehat{\pi}_j\overline{\psi}(\widehat{N}_{\alpha(j)}) = \widehat{\pi}_j(\widehat{N}_j) = A_5$, so $\widehat{N}_{\alpha(j)} \nsubseteq C_i$. Thus $\alpha(j) = i$.

9) Denote $\sigma^l = \sigma_{I_l}$. Prove that the diagram

$$\begin{tikzcd} P^1 \arrow[r, "\sigma^1"] \arrow[d, "\overline{\psi}"'] & k^{I_1} \arrow[d, "\alpha^*"] \\ P^2 \arrow[r, "\sigma^2"] & k^{I_2} \end{tikzcd} \tag{6}$$

commutes.

For $i \in I_l$ let D_i be the only subgroup of P^l of index 2 containing C_i. If $j \in I_2$, then $\overline{\psi}^{-1}(D_j)$ has index 2 in P^1 and contains $\overline{\psi}^{-1}(C_j) = C_{\alpha(j)}$, so $\overline{\psi}^{-1}(D_j) = D_{\alpha(j)}$. Further, let $g \in P^l$ and $v = \sigma^l(g) = (v_i \mid i \in I_l)$. Then $v_i = 0$ when $g \in D_i$ and $= 1$ when $g \notin D_i$. Now let $g \in P^1$, $h = \overline{\psi}(g)$, $u = \sigma^1(g) = (u_i \mid i \in I_1)$, and $v = \sigma^2(h) = (v_j \mid j \in I_2)$. Then $v_j = 0$ is equivalent to $h \in D_j$, which is equivalent to $g \in D_{\alpha(j)}$, which is, in turn, equivalent to $u_{\alpha(j)} = 0$. Thus we always have $v_j = u_{\alpha(j)}$. So $v = \alpha^*(u)$ and (6) commutes.

10) Finally, since the vertical arrows of diagram (6) are epimorphisms, it follows that α^* maps $\operatorname{Im}\sigma^1 = V_1$ onto $\operatorname{Im}\sigma^2 = V_2$, i.e., $\alpha^*(V_1) = V_2$. □

3.33. *The following statements are equivalent*:

(i) $R(I_1, V_1, W_1) = R_1 \cong R(I_2, V_2, W_2) = R_2$;
(ii) *there exists a bijection* $\alpha : I_1 \longrightarrow I_2$ *such that* $V_1 = \alpha^*(V_2)$ *and* $W_1 = \alpha^*(W_2)$.

Proof. (i) ⇒ (ii). It follows from 3.32 that there exists an injection $I_1 \longrightarrow I_2$ such that $\alpha^*(V_2) = V_1$ and $\alpha^*(W_2) \subseteq W_1$. Hence $|I_1| \leqslant |I_2|$. Then $|I_2| \leqslant |I_1|$ by the symmetry, whence $|I_1| = |I_2|$ and α is a bijection. So $\alpha^* : k^{I_2} \longrightarrow k^{I_1}$ is an isomorphism. So $\dim \alpha^*(W_2) = \dim W_2$, whence $\dim W_2 \leqslant \dim W_1$. By symmetry we have $\dim W_1 \leqslant \dim W_2$, so $\dim W_1 = \dim W_2$ and $\alpha^*(W_2) = W_1$.

(ii) $\Rightarrow$ (i). According to 3.30 there exists an epimorphism $\psi : Q(I_2, V_2) \longrightarrow Q(I_1, V_1)$ such that the diagram

$$\begin{tikzcd} k^{I_2} \arrow[r, "\alpha^*"] \arrow[d, "\nu_{I_2}"'] & k^{I_1} \arrow[d, "\nu_{I_1}"] \\ Q(I_2, V_2) \arrow[r, "\psi"] & Q(I_1, V_1) \end{tikzcd}$$

is commutative. Hence $\psi(K(W_2)) = \psi(\nu_{I_2}(W_2)) = \nu_{I_1}(\alpha^*(W_2)) = \nu_{I_1}(W_1) = K(W_1)$. So the rule $\overline{\psi}(gK(W_2)) = \psi(g)K(W_1)$ gives a well-defined map from $Q(I_2, V_2)/K(W_2) = R_2$ to $Q(I_1, V_1)/K(W_1) = R_1$, and it is easy to see that $\overline{\psi}$ is an epimorphism. So R_1 is a quotient of R_2. Next, since α is a bijection, it follows that $\alpha^* : k^{I_2} \longrightarrow k^{I_1}$ is an isomorphism, and $(\alpha^*)^{-1} = \beta^*$, where $\beta = \alpha^{-1}$. Now $\alpha^*(V_2) = V_1$ implies $\beta^*(V_1) = V_2$, and $\beta^*(W_1) = W_2$ similarly. So R_2 is a quotient of R_1 whence $R_1 \cong R_2$. $\square$

The proof of Theorem 2.4 Now we finally begin to prove Theorem 2.4.

If $V \subseteq k^I$ is a wide subspace, then $X = (I, V)$ is an object of the category $\mathcal{D}$. We sometimes denote $Q(I, V)$ and $R(I, V, W)$ by $Q(X)$ and $R(X, W)$, respectively. As we have already noted in Section 2, the correspondence $H : X \rightsquigarrow H(X) = k^I$ is an epimorphic $k\mathcal{D}$-module.

Let F and S be as in Theorem 2.4. Define first a map $\varkappa : F \longrightarrow S$. Let $\mathfrak{F} \in F$, i.e., let $\mathfrak{F}$ be a formation such that $\mathfrak{P} \subseteq \mathfrak{F} \subseteq \mathfrak{Q}$. Let $X = (I, V) \in \mathcal{D}$; note that ν_I maps $H(X)$ onto $Q(X)^{\mathfrak{P}} = \widehat{K}^I$ isomorphically. Further, $Q(X)^{\mathfrak{F}} \subseteq Q(X)^{\mathfrak{P}} = \nu_I(H(X))$. So $D(X) = \nu_I^{-1}(Q(X)^{\mathfrak{F}})$ is a subspace of $H(X)$.

3.34. *$D : X \rightsquigarrow D(X)$ is an epimorphic submodule of H.*

Proof. Let $X_1, X_2 \in \mathcal{D}$ and $\varphi \in \mathrm{Hom}_{\mathcal{D}}(X_1, X_2)$. We must prove that $\varphi_*(D(X_1)) = D(X_2)$.

We have $X_1 = (I_1, V_1)$, $X_2 = (I_2, V_2)$, and $\varphi = \mu_\alpha$, where $\alpha : I_2 \longrightarrow I_1$ is an injection such that $\alpha^*(V_1) = V_2$. By 3.30 there exists an epimorphism $\psi : Q(X_1) \longrightarrow Q(X_2)$ such that the diagram

$$\begin{tikzcd} k^{I_1} \arrow[r, "\alpha^*"] \arrow[d, "\nu_{I_1}"'] & k^{I_2} \arrow[d, "\nu_{I_2}"] \\ Q(X_1) \arrow[r, "\psi"] & Q(X_2) \end{tikzcd}$$

commutes. As $H(X_l) = k^{I_l}$ and $\alpha^* = \varphi_*$, we get a commutative diagram

$$\begin{tikzcd} H(X_1) \arrow[r, "\varphi_*"] \arrow[d, "\nu_{I_1}"'] & H(X_2) \arrow[d, "\nu_{I_2}"] \\ Q(X_1) \arrow[r, "\psi"] & Q(X_2). \end{tikzcd}$$

According to 3.12 we have $\psi(Q(X_1)^{\mathfrak{F}}) = Q(X_2)^{\mathfrak{F}}$. Hence $D(X_2) = \nu_{I_2}^{-1}(Q(X_2)^{\mathfrak{F}}) = \nu_{I_2}^{-1}(\psi(Q(X_1)^{\mathfrak{F}})) = \nu_{I_2}^{-1}(\psi(\nu_{I_1}(D(X_1)))) = \nu_{I_2}^{-1}(\nu_{I_2}(\varphi_*(D(X_1)))) = \varphi_*(D(X_1))$ (the last equality holds since ν_{I_2} is injective). $\square$

Define $\varkappa(\mathfrak{F}) = D$.

Now we define a map $\eta : S \longrightarrow F$. Let $D \in S$ be an epimorphic submodule of H. Let $\mathfrak{F} = \mathfrak{F}(D)$ be the class of all (up to isomorphism) groups of the form $R(X, W) \times U$, where $X \in \mathcal{D}$, $U \in \mathfrak{E}$, and $D(X) \subseteq W \subseteq H(X)$. Clearly, $\mathfrak{P} \subseteq \mathfrak{F} \subseteq \mathfrak{Q}$. To prove that $\mathfrak{F}$ is a formation we need a lemma.

Lemma 3.35. *Let p be a prime and G_1, G_2, G_3 be groups such that $G_i = X_i \times Y_i$. Suppose that $O_p(X_i) \subseteq O^p(X_i)$, $i = 1, 2, 3$, and all Y_i are p-groups. Then*

1) *if $f : G_1 \longrightarrow G_2$ is an epimorphism, then $f_1 = \mathrm{pr}_{X_2} \circ (f|_{X_1}) : X_1 \longrightarrow X_2$ is also an epimorphism*;
2) *if G_2 is a quotient of G_1, then X_2 is a quotient of X_1*;
3) *if G_3 is a subdirect product of G_1 and G_2, then X_3 is a subdirect product of X_1 and X_2.*

Proof. 1) Note first that if G is a group such that $O_p(G) \subseteq O^p(G)$ and if $X \trianglelefteq G$ is its normal subgroup such that $XO_p(G) = G$, then $X = G$. Indeed, $G/X = O_p(G)X/X \cong O_p(G)/O_p(G) \cap X$ is a p-group, so $X \supseteq O^p(G) \supseteq O_p(G)$, whence $G = XO_p(G) = X$.

Obviously, $f_2 = \mathrm{pr}_{X_2} \circ f$ is an epimorphism of G_1 onto X_2. Hence $X_2 = f_2(X_1)f_2(Y_1)$. Since $Y_1 \subseteq O_p(G_1)$, we have $f_2(Y_1) \subseteq O_p(X_2)$, whence $X_2 = f_2(X_1)O_p(X_2)$. Now $f_1(X_1) = f_2(X_1) = X_2$ from the preceding paragraph.

2) Trivial from 1).

3) There exist epimorphisms $\varphi_i : G_3 \longrightarrow G_i$, $i = 1, 2$, such that $\operatorname{Ker}\varphi_1 \cap \operatorname{Ker}\varphi_2 = 1$. By 1), $f_i = \mathrm{pr}_{X_i} \circ \varphi_i|_{X_3}$ is an epimorphism of X_3 onto X_i. It is sufficient to prove that $\operatorname{Ker} f_1 \cap \operatorname{Ker} f_2 = 1$.

Define $f : X_3 \longrightarrow G_1 \times G_2$ by $f(g) = (\varphi_1(g), \varphi_2(g))$. Clearly, f is an embedding. Next, $G_1 \times G_2 = X_1 \times Y_1 \times X_2 \times Y_2 = X \times Y$, where $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$. Obviously, $\operatorname{Ker} f_1 \cap \operatorname{Ker} f_2 = f^{-1}(Y)$. Denote $U = f^{-1}(Y)$.

As f is an embedding and $Y \subseteq O_p(G_1 \times G_2)$, we have $U \subseteq O_p(X_3)$. Next, $\mathrm{pr}_Y \circ f$ is a homomorphism from X_3 to a p-group, so it is trivial on $O_p(X_3)$ and thereby on U. That is, $(\mathrm{pr}_Y \circ f)(U) = 1$. Also $(\mathrm{pr}_X \circ f)(U) = \mathrm{pr}_X f(U) \subseteq \mathrm{pr}_X Y = 1$. Since the projections of $f(U)$ to both X and Y are trivial, we conclude that $f(U) = 1$, whence $U = 1$. □

3.36. *$\mathfrak{F}(D)$ is closed under taking quotients.*

Proof. Let $G \in \mathfrak{F}(D)$, then $G \cong B \times U$, where $B = R(X, W)$, $X = (I, V)$, $W \supseteq D(X)$, and $U \in \mathfrak{E}$. Suppose G_1 is a quotient for G, then $G \in \mathfrak{Q}$ and so $G_1 \cong B_1 \times U_1$, where $B_1 = R(I_1, V_1, W_1) = R(X_1, W_1)$, $U_1 \in \mathfrak{E}$. For $B = R(I, V, W)$ we have $O_2(B) = B^{\mathfrak{P}} \subseteq B^{\mathfrak{E}} = O^2(B)$ by 3.29, and similarly for B_1. By Lemma 3.35 B_1 is a quotient for B. Let $\psi : B \longrightarrow B_1$ be an epimorphism. By 3.32 there exists an embedding $\alpha : I_1 \longrightarrow I$ such that $\alpha^*(V) = V_1$ and $\alpha^*(W) \subseteq W_1$. The condition $\alpha^*(V) = V_1$ means that $\varphi = \mu_\alpha \in \mathrm{Hom}_{\mathcal{D}}(X, X_1)$. Then the condition $\alpha^*(W) \subseteq W_1$ means that $\varphi_*(W) \subseteq W_1$. Hence $W_1 \supseteq \varphi_*(W) \supseteq \varphi_*(D(X)) = D(X_1)$. Since $W_1 \supseteq D(X_1)$, it follows that $G_1 \in \mathfrak{F}(D)$. □

3.37. *$\mathfrak{F}(D)$ is closed under taking subdirect products.*

Proof. Let $G_1, G_2 \in \mathfrak{F}(D)$, and let G be their subdirect product. We may assume $G_l = B_l \times U_l$, where $B_l = R(I_l, V_l, W_l) = R(X_l, W_l)$, where $X_l = (I_l, V_l) \in \mathcal{D}$ and $W_l \supseteq D(X_l)$, and $U_l \in \mathfrak{E}$. As G is a subdirect product of $G_l \in \mathfrak{Q}$, we have $G \in \mathfrak{Q}$, so $G = B \times U$, $B = R(I, V, W) = R(X, W)$, and $U \in \mathfrak{E}$. Now it follows from Lemma 3.35, similarly to the argument in the proof of 3.36, that B is a subdirect product of B_1 and B_2. So there exist epimorphisms $\psi_l : B \longrightarrow B_l$ such that $\operatorname{Ker}\psi_1 \cap \operatorname{Ker}\psi_2 = 1$. By 3.32 there exist embeddings $\alpha_l : I_l \longrightarrow I$ such that $\alpha_l^*(V) = V_l$ and $\alpha_l^*(W) \subseteq W_l$ and, moreover, $T \cap \operatorname{Ker}\psi_l = K(W_l')/K(W)$, where $W_l' = (\alpha_l^*)^{-1}(W_l)$ and $T = T(B)$. Since $\operatorname{Ker}\psi_1 \cap \operatorname{Ker}\psi_2 = 1$, it follows that

$$1 = (T \cap \operatorname{Ker}\psi_1) \cap (T \cap \operatorname{Ker}\psi_2) = \big(K\big(W_1'\big)/K(W)\big) \cap \big(K\big(W_2'\big)/K(W)\big).$$

Observe that if $X, Y, Z \subseteq k^I$ are any three subspaces such that $X, Y \supseteq Z$, then $(K(X)/K(Z)) \cap (K(Y)/K(Z)) = K(X \cap Y)/K(Z)$. Thus the preceding equality implies $W_1' \cap W_2' = W$.

Since $\alpha_l^*(V) = V_l$, it follows that $\varphi_l = \mu_{\alpha_l}$ is a $\mathcal{D}$-morphism from X to X_l, and $(\varphi_l)_* : H(X) \longrightarrow H(X_l)$ is nothing else but $\alpha_l^* : k^I \longrightarrow k^{I_l}$. Thus, $W_l' = ((\varphi_l)_*)^{-1}(W_l)$. Since D is epimorphic, we have $(\varphi_l)_*(D(X)) = D(X_l) \subseteq W_l$, whence $W_l' = ((\varphi_l)_*)^{-1}(W_l) \supseteq D(X)$. So $W = W_1' \cap W_2' \supseteq D(X)$. This means just $G \in \mathfrak{F}(D)$. □

Define $\eta(D) = \mathfrak{F}(D)$.

Now we prove that $\varkappa$ and η are inverse bijections.

3.38. $\varkappa\eta = \mathrm{id}_S$.

Proof. Take arbitrary $D_1 \in S$, let $\mathfrak{F} = \mathfrak{F}(D_1) = \eta(D_1)$ and $D_2 = \varkappa(\mathfrak{F})$. We must prove that $D_1 = D_2$.

Let $X \in \mathcal{D}$. Then $Q(X)/K(D_1(X)) = R(X, D_1(X))$ lies in $\mathfrak{F}$ by the definition of $\mathfrak{F}(D_1)$. So $Q(X)^{\mathfrak{F}} \subseteq K(D_1(X))$, whence $D_2(X) = \nu_I^{-1}(Q(X)^{\mathfrak{F}}) \subseteq \nu_I^{-1}(K(D_1(X))) = D_1(X)$. Hence $D_2 \subseteq D_1$.

On the other hand, since $Q(X)^{\mathfrak{F}} = K(D_2(X))$, it follows that $R(X, D_2(X)) \in \mathfrak{F}$ for every $X \in \mathcal{D}$. So $R(X, D_2(X)) \cong R(Y, W) \times U$ for some $Y \in \mathcal{D}$ and $W \subseteq H(Y)$ such that $W \supseteq D_1(Y)$, and $U \in \mathfrak{E}$. In particular, there exists an epimorphism from $R(Y, W) \times U$ onto $R(X, D_2(X))$. Since both $G = R(Y, W)$ and $R(X, D_2(X))$ satisfy the condition $O_2(G) \subseteq O^2(G)$, it follows from Lemma 3.35 that there exists an epimorphism from $R(Y, W)$ onto $R(X, D_2(X))$. Hence $R(Y, W) \cong R(X, D_2(X))$ and $U = 1$. Now by 3.33 there exists a $\mathcal{D}$-isomorphism $\varphi : Y \longrightarrow X$ such that $\varphi_*(W) = D_2(X)$. As $W \supseteq D_1(Y)$ and D_1 is epimorphic, we see that $\varphi_*(W) \supseteq \varphi_*(D_1(Y)) = D_1(X)$, whence $D_2(X) \supseteq D_1(X)$ and $D_2 \supseteq D_1$. □

3.39. $\eta\varkappa = \mathrm{id}_F$.

Proof. Let $\mathfrak{L} \in F$, $D = \varkappa(\mathfrak{L})$, and $\mathfrak{L}_1 = \mathfrak{F}(D)$. We need to prove $\mathfrak{L}_1 = \mathfrak{L}$.

Let $G \in \mathfrak{L}$ and $G = R(X, W) \times U$. Then $R(X, W) \in \mathfrak{L}$, whence $Q(X)^{\mathfrak{L}} \subseteq K(W)$. But $Q(X)^{\mathfrak{L}} = K(D(X))$ by the definition of D, so $K(D(X)) \subseteq K(W)$, whence $D(X) \subseteq W$. Then $R(X, W) \in \mathfrak{F}(D)$ from the definition of $\mathfrak{F}(D)$, whence $G \in \mathfrak{F}(D) = \mathfrak{L}_1$. So $\mathfrak{L} \subseteq \mathfrak{L}_1$.

Conversely, let $G \in \mathfrak{L}_1$. Then $G = R(X, W) \times U$, where $X \in \mathcal{D}$, $W \supseteq D(X)$, and $U \in \mathfrak{E}$. Next, $R(X, W)$ is a quotient for $R(X, D(X)) = Q(X)/K(D(X)) = Q(X)/Q(X)^{\mathfrak{L}} \in \mathfrak{L}$. So $R(X, W) \in \mathfrak{L}$, whence $G \in \mathfrak{L}$ and $\mathfrak{L}_1 \subseteq \mathfrak{L}$. □

It is evident that both $\varkappa$ and η invert (non-strict) inclusion, that is $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ implies $\varkappa(\mathfrak{F}_1) \supseteq \varkappa(\mathfrak{F}_2)$ for any $\mathfrak{F}_1, \mathfrak{F}_2 \in F$, and similarly for η.

Lemma 3.40. *Let X and Y be posets, and suppose both maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ are monotonic, and are bijections, inverse to each other. Then both f and g are inverse poset isomorphisms or antiisomorphisms.*

It follows immediately from the lemma that $\varkappa$ and η are poset antiisomorphisms, inverse to each other. This finishes the proof of Theorem 2.4.

4. The $k\mathcal{D}$-submodules of H

We warn the reader that the notation in this section is, as a rule, independent of the previous section.

The submodule $\overline{D}$ Let $X = (I, V) \in \mathcal{D}$. Define an equivalence on I by $i \sim j$ if $v_i = v_j$ for all $v \in V$.

4.1. *Let $\mu_f \in \mathrm{Hom}_{\mathcal{D}}((I, V), (J, U))$ and $i, j \in J$. Then $i \sim j$ if and only if $f(i) \sim f(j)$.*

Proof. The condition $f(i) \sim f(j)$ means that $v_{f(i)} = v_{f(j)}$ for every $v \in V$. But $v_{f(i)} = f^*(v)_i$ and $v_{f(j)} = f^*(v)_j$, so the latter condition is equivalent to $f^*(v)_i = f^*(v)_j$ for every $v \in V$, i.e., to $u_i = u_j$

for every $u \in f^*(V)$. But $f^*(V) = U$, so the latter condition is equivalent to $u_i = u_j$ for all $u \in U$, i.e., is equivalent to the condition $i \sim j$. $\square$

For $X = (I, V) \in \mathcal{D}$ let $\overline{D}(X)$ be the subspace of $H(X) = k^I$ consisting of all arrays $(\lambda_i \mid i \in I)$ such that $\lambda_i = \lambda_j$ whenever i and j are equivalent.

4.2. *$\overline{D}$ is an epimorphic submodule of H.*

Proof. Let $X = (I, V)$, $Y = (J, U)$, $\varphi = \mu_f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$, and $n \in \overline{D}(X)$. Let $i, j \in J$ and $i \sim j$. Then $f^*(n)_i = n_{f(i)} = n_{f(j)} = f^*(n)_j$; the second equality follows from the definition of $\overline{D}(X)$ and the fact that $f(i) \sim f(j)$. Hence $\varphi_*(n) = f^*(n) \in \overline{D}(Y)$, so $\overline{D}$ is a $k\mathcal{D}$-submodule of H.

Show that $\overline{D}$ is epimorphic. Let $n \in \overline{D}(Y)$. Define $m \in H(X)$. Put $m_{f(i)} = n_i$ when $i \in J$. Note that if $f(i) \sim f(j)$, then $i \sim j$, whence $m_{f(i)} = m_{f(j)}$. Next, if $l \in I \setminus f(J)$ and $l \sim f(j)$ for some $j \in J$, then we put $m_l = m_{f(j)}$. Clearly, m_l is well defined. Finally, if $l \in I$ is not equivalent to any $f(j)$, then we put $m_l = 0$. Now it is easy to see that $m \in \overline{D}(X)$ and $\varphi_*(m) = n$. So $\overline{D}$ is epimorphic. $\square$

4.3. *Let $L \subseteq H$ be an epimorphic submodule such that $L \nsubseteq \overline{D}$. Then $L = H$.*

Proof. First note the following. Let $X = (I, V)$ be an arbitrary object of $\mathcal{D}$ and π be a permutation on I preserving each equivalence class. Then μ_π is a $\mathcal{D}$-automorphism of X.

By e_i, $i \in I$, we denote the standard basis vectors of k^I, as in Section 3.

Consider the object $X_0 = (I_0, V_0) \in \mathcal{D}$, where $I_0 = \{1, 2\}$ and $V_0 \subseteq k^{I_0}$ is the one-dimensional subspace spanned by $(1, 1) = e_1 + e_2$. It is easy to show that for any $X = (I, V) \in \mathcal{D}$ the morphisms from X to X_0 are precisely μ_f, where $f : I_0 \longrightarrow I$ is an injection such that $f(1) \sim f(2)$.

Show that $L(X_0) = k^{I_0}$ (i.e., the space of all pairs $(x, y) \in k^2$). Since $L \nsubseteq \overline{D}$, there are $X = (I, V) \in \mathcal{D}$, $m \in L(X)$ and $i, j \in I$ such that $i \sim j$ but $m_i \neq m_j$. Define $f : I_0 \longrightarrow I$ by $f(1) = i$, $f(2) = j$. Then $\mu_f \in \mathrm{Hom}_{\mathcal{D}}(X, X_0)$. As L is epimorphic, we have $L(X_0) = (\mu_f)_*(L(X)) \ni (\mu_f)_*(m) = f^*(m) = m_i e_1 + m_j e_2$, which we denote by n. Similarly, let $f' : I_0 \longrightarrow I$, $f'(1) = j$, $f'(2) = i$. Then $L(X_0)$ contains also $(\mu_{f'})_*(m) = m_j e_1 + m_i e_2 = n'$. As $m_i \neq m_j$, we have $\{n, n'\} = \{e_1, e_2\}$, whence $L(X_0) = k^{I_0}$.

Further, let $X = (I, V) \in \mathcal{D}$ be an arbitrary object, $i, j \in I$, $i \sim j$, and $i \neq j$. Prove that $e_i + e_j \in L(X)$. Consider $\theta = \mu_f \in \mathrm{Hom}_{\mathcal{D}}(X, X_0)$, where $f(1) = i$, $f(2) = j$. Since L is epimorphic, we have $\theta_*(L(X)) = L(X_0) \ni e_1$. It follows that $L(X)$ contains an element $m = (m_l \mid l \in I)$ such that $m_i = 1$, $m_j = 0$. That is, $m = e_i + m'$, where m' is a linear combination of e_l, $l \neq i, j$. Next, let $t = (ij)$ be the transposition on I. Then $\rho = \mu_t \in \mathrm{Aut}_{\mathcal{D}}(X)$. Since ρ_* interchanges e_i and e_j and fixes e_l for all $l \neq i, j$, it follows that $\rho_*(m) = e_j + m'$, whence $L(X) \ni m + \rho_*(m) = e_i + e_j$.

Now we consider arbitrary $X = (I, V) \in \mathcal{D}$ and take any $i \in I$. Take a symbol $p \notin I$ and define $I_1 = I \cup \{p\}$. Let $V_1 \subseteq k^{I_1}$ be the space of all arrays $v = (v_l \mid l \in I_1)$ such that $\pi_I(v) = (v_l \mid l \in I)$ is in V, and $v_p = v_i$. Clearly V_1 is wide, so $X_1 = (I_1, V_1)$ is an object of $\mathcal{D}$, and $i \sim p$ in I_1. Let $g : I \longrightarrow I_1$ be the obvious injection. Then $g^*(V_1) = V$, whence $\psi = \mu_g \in \mathrm{Hom}_{\mathcal{D}}(X_1, X)$. By the preceding paragraph, $e_i + e_p \in L(X_1)$. So $e_i = g^*(e_i + e_p) = \psi_*(e_i + e_p) \in \psi_*(L(X_1)) = L(X)$. Thus, $L(X)$ contains all basis elements of $H(X)$, and $L(X) = H(X)$. $\square$

Restriction to subcategories Now we begin to study the epimorphic $k\mathcal{D}$-submodules of $\overline{D}$.

For a field k, a category $\mathcal{C}$ and a $k\mathcal{C}$-module M we denote by $S_{k\mathcal{C}}(M)$ the poset of all epimorphic $k\mathcal{C}$-submodules of M.

We first show that it is sufficient to consider smaller subcategory. We begin with a lemma.

Lemma 4.4. *Let k be an arbitrary field, $\mathcal{C}$ be a category, $\mathcal{C}' \subseteq \mathcal{C}$ be its full subcategory, M be an epimorphic $k\mathcal{C}$-module, and $M' = M|_{\mathcal{C}'}$. Let $S = S_{k\mathcal{C}}(M)$, $S' = S_{k\mathcal{C}'}(M')$, and $\varkappa : S \longrightarrow S'$ be the restriction to $\mathcal{C}'$, i.e., $\varkappa(L) = L|_{\mathcal{C}'}$. We call a morphism $\varphi : X \longrightarrow X'$* a contracting morphism (*with respect to M*) *if $X' \in \mathcal{C}'$ and $\varphi_* : M(X) \longrightarrow M(X')$ is an isomorphism of spaces. Suppose the following conditions are satisfied*:

(1) *all endomorphisms in $\mathcal{C}$ are automorphisms*;

(2) *for every $X \in \mathcal{C}$ there exists a contracting morphism $\varphi : X \longrightarrow X'$;*
(3) *if $X, Y \in \mathcal{C}$, $X', Y' \in \mathcal{C}'$, $\alpha \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$, and the morphisms $\varphi_X : X \longrightarrow X'$ and $\varphi_Y : Y \longrightarrow Y'$ are contracting, then there exists a commutative diagram of the form*

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\varphi_X"'] & X \arrow[r, "\alpha"] & Y \arrow[d, "\varphi_Y"] \\ X' \arrow[rr, "\beta"] & & Y' \end{tikzcd} \tag{7}$$

such that γ_* is trivial on $M(X)$.

Then $\varkappa$ is a poset isomorphism.

Proof. Let $L \in S$, $L' = L|_{\mathcal{C}'}$, and $X \in \mathcal{C}$. Choose a contracting morphism $\varphi : X \longrightarrow X'$. Then $\varphi_* : M(X) \longrightarrow M(X')$ is an isomorphism, and $\varphi_*(L(X)) = L(X') = L'(X')$. Hence $L(X) = \varphi_*^{-1}(L'(X'))$. So L can be uniquely recovered from L', i.e., $\varkappa$ is injective. It is also clear that $L_1 \subseteq L_2$ if and only if $L'_1 \subseteq L'_2$. So it suffices to prove that $\varkappa$ is surjective or, equivalently, to prove that the rule $L(X) = \varphi_*^{-1}(L'(X'))$ gives a well-defined epimorphic submodule of M, for every $L' \in S'$.

First prove that the space $L(X)$ is well defined for every $X \in \mathcal{C}$. That is, we prove that

$$\left(\varphi'_*\right)^{-1} L'\left(X'\right) = \left(\varphi''_*\right)^{-1} L'\left(X''\right) \tag{8}$$

for any two contracting morphisms $\varphi' : X \longrightarrow X'$ and $\varphi'' : X \longrightarrow X''$. It follows from the condition (3) with $Y = X$ and $\alpha = \mathrm{id}_X$ that there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\varphi'"'] & X \arrow[d, "\varphi''"] \\ X' \arrow[r, "\beta"] & X'' \end{tikzcd}$$

such that $\gamma_* = \mathrm{id}_{M(X)}$. This diagram leads to the diagram

$$\begin{tikzcd} M(X) \arrow[r, equal] \arrow[d, "\varphi'_*"'] & M(X) \arrow[d, "\varphi''_*"] \\ M(X') \arrow[r, "\beta_*"] & M(X''). \end{tikzcd}$$

Now the equality (8) follows from the fact that φ'_* and φ''_* are isomorphisms and $L'(X'') = \beta_*(L'(X'))$ as L' is epimorphic.

It remains to check that $L : X \rightsquigarrow L(X)$ is an epimorphic submodule. Let $\alpha : X \longrightarrow Y$ be a $\mathcal{C}$-morphism. Take contracting morphisms $\varphi_X : X \longrightarrow X'$ and $\varphi_Y : Y \longrightarrow Y'$ and consider a commutative diagram as in (3). It leads to the diagram

$$\begin{tikzcd} M(X) \arrow[r, equal] \arrow[d, "(\varphi_X)_*"'] & M(X) \arrow[r, "\alpha_*"] & M(Y) \arrow[d, "(\varphi_Y)_*"] \\ M(X') \arrow[rr, "\beta_*"] & & M(Y'). \end{tikzcd}$$

Since $(\varphi_X)_*$ and $(\varphi_Y)_*$ are isomorphisms, it follows that $\alpha_* = (\varphi_Y)_*^{-1}\beta_*(\varphi_X)_*$. Next, $(\varphi_X)_*L(X) = L'(X')$ from the definition of $L(X)$ and $\beta_*L'(X') = L'(Y')$ (because L' is epimorphic), and $(\varphi_Y)_*^{-1}L'(Y') = L(Y)$. Therefore $\alpha_*L(X) = L(Y)$, i.e., L is an epimorphic submodule. □

Let $\mathcal{D}_1 \subseteq \mathcal{D}$ be the full subcategory consisting of all $X = (I, V)$ such that no two distinct $i \neq j \in I$ are equivalent. It is clear that $X \in \mathcal{D}_1$ if and only if $H(X) = \overline{D}(X)$.

4.5. *$\mathcal{C} = \mathcal{D}$, $\mathcal{C}' = \mathcal{D}_1$, and $M = \overline{D}$ satisfy the hypothesis of Lemma 4.4.*

Proof. It was already observed in 2.2 that all endomorphisms in $\mathcal{D}$ are actually automorphisms.

Let $X = (I, V)$, let $I' \subseteq I$ be a subset that meets each equivalency class in a point, and let $f : I' \longrightarrow I$ be the (tautological) injection. Set $V' = f^*(V) = \pi_{I'}(V)$. It is easy to see that $X' = (I', V') \in \mathcal{D}_1$ and that $\varphi = \mu_f : X \longrightarrow X'$ is a $\mathcal{D}$-morphism such that φ_* maps isomorphically $\overline{D}(X)$ onto $\overline{D}(X')$. (Note by the way that $\overline{D}(X') = H(X') = k^{I'}$.) That is, $\varphi : X \longrightarrow X'$ is a contracting (with respect to $\overline{D}$) morphism, and we obtain (2). Note the following: a $\mathcal{D}$-morphism $\varphi = \mu_f : (I, V) \longrightarrow (I_1, V_1)$ is contracting precisely when $f(I_1)$ meets each equivalency class in I in a point.

Finally, let X, Y, X', Y', α, φ_X, φ_Y be as in (3); $X = (I, V)$, $Y = (J, U)$, $X' = (I', V')$, $Y' = (J', U')$, $\alpha = \mu_a$, where $a : J \longrightarrow I$ and $a^*(V) = U$. Moreover, $\varphi_X = \mu_{f_X}$ and $\varphi_Y = \mu_{f_Y}$, where $f_X : I' \longrightarrow I$ and $f_Y : J' \longrightarrow J$ are some injections such that $f_X(I')$ meets each equivalency class in I in a point and similarly for f_Y.

Recall that $a(i) \sim a(j)$ if and only if $i \sim j$. Consider the composition map $af_Y : J' \xrightarrow{f_Y} J \xrightarrow{a} I$. Any equivalency class in I contains a unique element of the form $f_X(x)$, where $x \in I'$. In particular, for any $t \in J'$ there exists a unique element $b(t) \in I'$ such that $f_X(b(t)) \sim a(f_Y(t))$. Next, if $u \neq t \in J'$, then $f_Y(u) \nsim f_Y(t)$ and so $a(f_Y(u)) \nsim a(f_Y(t))$, whence $f_X(b(u)) \nsim f_X(b(t))$ and finally $b(u) \neq b(t)$. So $b : J' \longrightarrow I'$ is an injection. Moreover, since $f_X(b(t))$ and $a(f_Y(t))$ are in the same equivalency class and this class does not contain other $f_X(b(u))$ or $a(f_Y(u))$, we see that there exists a permutation g of I such that g preserves equivalency classes and $f_X(b(t)) = g(a(f_Y(t)))$ for all $t \in J'$. Thus, we have a commutative diagram of sets

$$\begin{tikzcd} I & I \arrow[l, "g"'] & J \arrow[l, "a"'] \\ I' \arrow[u, "f_X"] & & J'. \arrow[ll, "b"'] \arrow[u, "f_Y"'] \end{tikzcd} \tag{9}$$

We show that $\gamma = \mu_g$ and $\beta = \mu_b$ satisfy conditions of (3).

First, g^* acts identically on $\overline{D}(X)$, because g preserves equivalency classes. In particular $g^*(V) = V$, so $\gamma = \mu_g \in \mathrm{Aut}_{\mathcal{D}}(X)$, and $\gamma_* = g^*$ is trivial on $\overline{D}(X)$.

Further, from diagram (9) we obtain (dual) diagram

$$\begin{tikzcd} k^I \arrow[r, "g^*"] \arrow[d, "f_X^*"'] & k^I \arrow[r, "a^*"] & k^J \arrow[d, "f_Y^*"] \\ k^{I'} \arrow[rr, "b^*"] & & k^{J'}. \end{tikzcd} \tag{10}$$

Since α, φ_X and φ_Y are $\mathcal{D}$-morphisms, we see that $a^*(V) = U$, $f_X^*(V) = V'$ and $f_Y^*(U) = U'$. Hence $b^*(V') = b^*(f_X^*(V)) = f_Y^*a^*g^*(V) = f_Y^*a^*(V) = f_Y^*(U) = U'$. So $\beta = \mu_b$ is a $\mathcal{D}$-morphism from X' to Y'. Finally, diagram (7) is commutative as (9) is commutative. □

Corollary 4.6. *The posets of all epimorphic $k\mathcal{D}$-submodules of $\overline{D}$ and all epimorphic $k\mathcal{D}_1$-submodules of $H_1 = H|_{\mathcal{D}_1}$ are isomorphic.*

Lemma 4.7. *Let $\mathcal{C}$ be a category, $\mathcal{C}' \subseteq \mathcal{C}$ be its full subcategory, k be a field, and M be an epimorphic $k\mathcal{C}$-module. Let $M' = M|_{\mathcal{C}'}$, $S = S_{k\mathcal{C}}(M)$, and $S' = S_{k\mathcal{C}'}(M')$. Let E be some subset of the set of all morphisms $f : X \longrightarrow Y$ whose domain is $X \in \mathcal{C}'$. We call the elements of E* the distinguished morphisms*.*

Suppose that for every $X \in \mathcal{C}$ there exists a distinguished morphism $f : X' \longrightarrow X$ and that for every diagram of the form

$$\begin{tikzcd} X' \arrow[d, "g_X"'] & Y' \arrow[d, "g_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{11}$$

such that g_X and g_Y are distinguished there exists a morphism $f' : X' \longrightarrow Y'$ that makes this diagram commute.

Then the restriction map $\varkappa : S \longrightarrow S'$ defined by $\varkappa(L) = L|_{\mathcal{C}'}$ is a poset isomorphism.

Proof. Note that $L \in S$ can be uniquely recovered from $L' = L|_{\mathcal{C}'}$. Indeed, if $X \in \mathcal{C}$, $X' \in \mathcal{C}'$, and $f : X' \longrightarrow X$ is a distinguished morphism, then $L(X) = f_*(L(X')) = f_*(L'(X'))$. So $\varkappa$ is injective. Moreover, $L_1 \subseteq L_2$ if and only if $\varkappa(L_1) \subseteq \varkappa(L_2)$. So it suffices to prove that $\varkappa$ is surjective. To do this it suffices, in turn, to show that for every $L' \in S'$ the rule $L(X) = f_*(L'(X'))$, where $f : X' \longrightarrow X$ is a distinguished morphism, gives a well-defined epimorphic $k\mathcal{C}$-module.

First check that for a given $X \in \mathcal{C}$ the subspace $L(X)$ is well defined. That is, we check that $f'_*(L'(X')) = f''_*(L'(X''))$ for any pair of distinguished morphisms $f' : X' \longrightarrow X$, $f'' : X'' \longrightarrow X$. In diagram (11) take $Y = X$, $Y' = X''$, $g_X = f'$, $g_Y = f''$, and $f = \mathrm{id}_X$. We see that there exists a morphism $h : X' \longrightarrow X''$ such that $f' = f''h$. Since L' is epimorphic, it follows that $h_*(L'(X')) = L'(X'')$, whence $f'_*(L'(X')) = (f''h)_*(L'(X')) = f''_*h_*(L'(X')) = f''_*(L'(X''))$. Thus $L(X)$ is well defined.

Now we prove that $L : X \rightsquigarrow L(X)$ is an epimorphic submodule. Let $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Take distinguished morphisms $g_X : X' \longrightarrow X$ and $g_Y : Y' \longrightarrow Y$. By the hypothesis there exists $h : X' \longrightarrow Y'$ such that $fg_X = g_Y h$. Hence $f_*(L(X)) = f_*((g_X)_*(L'(X'))) = (fg_X)_*(L'(X')) = (g_Y h)_*(L'(X')) = (g_Y)_* h_*(L'(X')) = (g_Y)_*(L'(Y')) = L(Y)$. $\square$

Let $V \in \mathbf{fVect}_k$ be a space over k of dimension $1 \leqslant d < \infty$, and let $I(V) = V^* \setminus \{0\}$ be the set of all nontrivial linear functions on V. Define $\alpha = \alpha_V : V \longrightarrow k^{I(V)}$ by $\alpha(v) = (w_l \mid l \in I(V))$, where $w_l = l(v)$. It is easy to see that α is injective and $V' = \alpha(V)$ is wide, so $K(V) = (I(V), V')$ is an object of $\mathcal{D}$. Next, for any $l \neq m \in I(V)$ there exists v such that $l(v) \neq m(v)$, so $l \nsim m$. So $K(V) \in \mathcal{D}_1$.

Introduce a notation. Let $X = (I, V)$, $Y = (J, U) \in \mathcal{D}$, and $\varphi = \mu_f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$. Then $f^*(V) = U$, hence $f^*|_V = \varphi_*|_V$ is an epimorphism from V onto U. We denote $\widehat{\varphi} = \varphi_*|_V$. It is easy to see that always $\widehat{\varphi\psi} = \widehat{\varphi}\widehat{\psi}$.

4.8. *Let $X = (I, V) \in \mathcal{D}_1$, $U \in \mathbf{fVect}_k$, $K(U) = (I(U), U')$, and $\dim U \geqslant \dim V$. Then for any epimorphism $\beta : U' \longrightarrow V$ there exists a unique $\mathcal{D}_1$-morphism $\varphi : K(U) \longrightarrow X$ such that $\widehat{\varphi} = \beta$.*

Proof. First note the following. Let I and J be some finite sets, $V \subseteq k^I$ be a subspace, $f : J \longrightarrow I$ be a set map and $\gamma : V \longrightarrow k^J$ be a linear map. Then the following conditions are equivalent: (1) $f^*|_V = \gamma$, and (2) $\gamma(v)_j = v_{f(j)}$ for every $j \in J$, $v \in V$.

Indeed, we have $f^*(v)_j = v_{f(j)}$ for any $v \in k^I$ and $j \in J$ by the definition of f^*. Now the condition (1) is equivalent to the condition that $f^*(v)_j = \gamma(v)_j$ for any $v \in V$ and $j \in J$, which is equivalent to the condition $\gamma(v)_j = v_{f(j)}$ for any $v \in V$ and $j \in J$, but the latter is precisely the condition (2).

Now we prove the statement.

Let $\alpha = \alpha_U : U \longrightarrow k^{I(U)}$ be the map from the definition of $K(U)$. The composite map $\beta\alpha$ is an epimorphism of U onto V. For $i \in I$ define $l_i : U \longrightarrow k$ by $l_i(u) = \beta\alpha(u)_i$. Clearly, l_i is a nontrivial linear function on U. Next, let $i \neq j$. There exists an element $v \in V$ such that $v_i \neq v_j$. Let $u \in U$ be an element such that $\beta\alpha(u) = v$. Then $l_i(u) = \beta\alpha(u)_i = v_i \neq v_j = l_j(u)$. So $l_i \neq l_j$. Thus, the rule $i \longmapsto l_i$ defines an injection $f_1 : I \longrightarrow I(U)$.

Let $f : I \longrightarrow I(U)$ be an injection. Then the condition that $\varphi = \mu_f \in \mathrm{Hom}_{\mathcal{D}}(K(U), X)$ and $\widehat{\varphi} = \beta$ is equivalent to $f^*|_{U'} = \beta$, which is equivalent, by the observation in the beginning of the proof, to the fact that $\beta(u')_i = u'_{f(i)}$ for all $u' \in U'$ and $i \in I$. As α maps U isomorphically onto U', the latter is equivalent to $\beta\alpha(u)_i = \alpha(u)_{f(i)}$ for all $u \in U$ and $i \in I$. Since $\beta\alpha(u)_i = l_i(u)$ and $\alpha(u)_{f(i)} = f(i)(u)$, the latter is equivalent to $l_i(u) = f(i)(u)$ for all u and i, i.e., equivalent to $l_i = f(i)$ for all i. Thus, the condition "$\varphi = \mu_f \in \mathrm{Hom}_{\mathcal{D}}(K(U), X)$ and $\widehat{\varphi} = \beta$" is equivalent to the equality $f = f_1$, which finishes the proof. $\square$

For each $d \in \mathbf{N}$ fix a d-dimensional space $U_d \in \mathbf{fVect}_k$, and define $K_d = K(U_d)$. Let $\mathcal{D}_2 \subset \mathcal{D}_1$ be the full subcategory that consists of all K_d, and $H_2 = H_1|_{\mathcal{D}_2} = H|_{\mathcal{D}_2}$.

4.9. *Let $\mathcal{C} = \mathcal{D}_1$, $\mathcal{C}' = \mathcal{D}_2$, and let E be the class of all $\mathcal{D}_1$-morphisms of the form $\varphi : K(U_d) \longrightarrow X = (I, V)$ such that* $\dim V = d$. *Then $\mathcal{C}$, $\mathcal{C}'$, and E satisfy the hypothesis of Lemma 4.7.*

Proof. Let $X = (I, V) \in \mathcal{D}_1$ and $d = \dim V$. Since $\dim U'_d = d$, there exists an epimorphism $f : U'_d \longrightarrow V$. By 4.8 there exists $\varphi : K(U_d) \longrightarrow X$ such that $\widehat{\varphi} = f$. That is, there exist $X' \in \mathcal{C}'$ and a distinguished morphism $\varphi : X' \longrightarrow X$.

Further, consider a diagram of the form (11). We have $X = (I_1, V_1)$, $Y = (I_2, V_2)$, $X' = K(U_{d_1})$, and $Y' = K(U_{d_2})$, where $d_i = \dim V_i$. Since $\dim U'_{d_i} = d_i = \dim V_i$, it follows that $\widehat{g}_X : U'_{d_1} \longrightarrow V_1$ is a space isomorphism, and similarly for $\widehat{g}_Y$. Since $\widehat{f}$ is an epimorphism of V_1 onto V_2, it follows that there exists an epimorphism $\alpha : U'_{d_1} \longrightarrow U'_{d_2}$ such that $\widehat{f}\,\widehat{g}_X = \widehat{g}_Y\alpha$. Let $\psi : X' \longrightarrow Y'$ be the unique morphism such that $\widehat{\psi} = \alpha$. Then $\widehat{g_Y\psi} = \widehat{g}_Y\widehat{\psi} = \widehat{g}_Y\alpha = \widehat{f}\,\widehat{g}_X = \widehat{fg_X}$. So both $\zeta = g_Y\psi$ and $\xi = fg_X$ are morphisms from $K(U_{d_1})$ to Y such that $\widehat{\zeta} = \widehat{\xi}$, whence $\zeta = \xi$ by 4.8. Thus $g_Y\psi = fg_X$, which proves the second condition of Lemma 4.7. $\square$

Corollary 4.10. *The posets $S_{k\mathcal{D}_1}(H_1)$ and $S_{k\mathcal{D}_2}(H_2)$ are isomorphic.*

The category $\mathcal{V}$ We need a description of $\mathcal{D}$-morphisms from $K(U)$ to $K(V)$, where $U, V \in \mathbf{fVect}_k$. If $U, V \in \mathbf{fVect}_k$ and $\dim U \geqslant \dim V$, and if $f : U \longrightarrow V$ is an epimorphism, then f induces an injection $t(f) : I(V) \longrightarrow I(U)$, which is the restriction to $I(V)$ of the usual dual map $f^* : V^* \longrightarrow U^*$.

4.11. 1) *If* $\dim U < \dim V$, *then* $\mathrm{Hom}_{\mathcal{D}}(K(U), K(V)) = \emptyset$.

2) *If* $\dim U \geqslant \dim V$ *and $f : U \longrightarrow V$ is an epimorphism, then $\gamma(f) = \mu_{t(f)}$ is a $\mathcal{D}$-morphism from $K(U)$ to $K(V)$.*

3) *Any element of* $\mathrm{Hom}_{\mathcal{D}}(K(U), K(V))$ *is of the form $\gamma(f)$ for a unique epimorphism f.*

Proof. 1) Clear, because in this case $\dim U' < \dim V'$ and so there is no an epimorphism from U' onto V'.

2) It is sufficient to check $t(f)^*(U') = V'$. Let $u \in U$ and $j \in I(V) = V^* \setminus \{0\}$. Then $(t(f)^*\alpha_U(u))_j = \alpha_U(u)_{t(f)(j)} = (t(f)(j))(u) = (f^*(j))(u) = j(f(u)) = \alpha_V(f(u))_j$. Since this is true for all $j \in I(V)$, it follows that $t(f)^*\alpha_U(u) = \alpha_V(f(u))$ for all $u \in U$. So $t(f)^*(U') = t(f)^*\alpha_U(U) = \alpha_V(f(U)) = \alpha_V(V) = V'$, as required.

3) Let $f : U \longrightarrow V$ be an epimorphism as earlier, and $g = \gamma(f) = \mu_{t(f)}$. Then $\widehat{g} = t(f)^*|_{U'}$. It follows from the formula $t(f)^*\alpha_U(u) = \alpha_V(f(u))$ and the fact that α_U and α_V are isomorphisms of U onto U' and V onto V', respectively, that $\alpha_V f\alpha_U^{-1}(\alpha_U(u)) = \alpha_V f(u) = t(f)^*\alpha_U(u)$ for every $u \in U$, so $\alpha_V f\alpha_U^{-1} = t(f)^*|_{U'}$. So $\widehat{\gamma(f)} = \widehat{g} = t(f)^*|_{U'} = \alpha_V f\alpha_U^{-1}$.

Now let $\theta \in \mathrm{Hom}_{\mathcal{D}}(K(U), K(V))$. Then $\widehat{\theta}$ is an epimorphism of U' onto V'. Since $\alpha_U : U \longrightarrow U'$ and $\alpha_V : V \longrightarrow V'$ are isomorphisms, it follows that $\widehat{\theta} = \alpha_V f\alpha_U^{-1}$ for appropriate epimorphism $f : U \longrightarrow V$. I.e., $\widehat{\theta} = \widehat{\gamma(f)}$. But by 4.8 for any epimorphism $h : U' \longrightarrow V'$ there exists a unique $\varphi \in \mathrm{Hom}_{\mathcal{D}}(K(U), K(V))$ such that $\widehat{\varphi} = h$. Hence $\theta = \gamma(f)$.

Finally, let $f_1, f_2 : U \longrightarrow V$ be different epimorphisms. Then the dual maps $f_1^*, f_2^* : V^* \longrightarrow U^*$ are different also, so $t(f_1) \neq t(f_2)$ and $\gamma(f_1) \neq \gamma(f_2)$. $\square$

Let $\mathcal{V}$ be the category whose objects are the spaces U_d, $d \geqslant 1$, and the morphisms are all vector space epimorphisms of these spaces.

For $V \in \mathbf{fVect}_k$ put $Q'(V) = k^{I(V)}$. If $f : U \longrightarrow V$ is an epimorphism, then $t(f) : I(V) \longrightarrow I(U)$ is an injection, which induces the epimorphism $Q'(f) : Q'(U) \longrightarrow Q'(V)$. It is easy to see that if $f : U \longrightarrow V$ and $g : V \longrightarrow W$ are epimorphisms, then $Q'(gf) = Q'(g)Q'(f)$. Therefore $Q = Q'|_{\mathcal{V}} : U_d \rightsquigarrow Q(U_d) = Q'(U_d)$ is an epimorphic $k\mathcal{V}$-module.

4.12. $S_{k\mathcal{D}_2}(H_2) \cong S_{k\mathcal{V}}(Q)$.

Proof. First of all we may identify $H(K(U_d)) = k^{I(U_d)} = Q(U_d)$. Next, a $k\mathcal{D}_2$-submodule of H_2 is, by definition, a collection of subspaces $M_d \subseteq H(K(U_d))$ such that $\varphi_*(M_d) = M_e$ for every morphism $\varphi \in \mathrm{Hom}_{\mathcal{D}}(K(U_d), K(U_e))$. By 4.11, φ is precisely $\mu_{t(f)}$ for some epimorphism $f : U_d \longrightarrow U_e$. As $\varphi_* = t(f)^*$, the condition $\varphi_*(M_d) = M_e$ means that $t(f)^*(M_d) = M_e$ for every epimorphism $f : U_d \longrightarrow U_e$. But $t(f)^*$ is nothing else but $Q(f)$. That is, a collection of subspaces $\{M_d \mid d \geqslant 1\}$ is a $k\mathcal{D}_2$-submodule of H_2 if and only if $Q(f)M_d = M_e$ for every epimorphism $f : U_d \longrightarrow U_e$, that is, when $\{M_d \mid d \geqslant 1\}$ is an epimorphic $k\mathcal{V}$-submodule of Q. $\square$

For a vector space $V \in \mathbf{fVect}_k$ (over arbitrary field) let $S^*(V)$ and $\Lambda^*(V)$ be the symmetric and the exterior algebra of V, respectively. It is well known that $V \rightsquigarrow S^*(V)$ and $V \rightsquigarrow \Lambda^*(V)$ are functors (from $\mathbf{fVect}_k$ to the category of commutative associative algebras with unit, and the category of finite dimensional associative algebras with unit, respectively). In particular, $V \rightsquigarrow S^*(V)$ and $V \rightsquigarrow \Lambda^*(V)$ are $k\mathbf{fVect}_k$-modules (the first of them is infinite dimensional). Next, $GL(V)$ acts on both $S^*(V)$ and $\Lambda^*(V)$. The subspaces $S^d(V)$ and $\Lambda^d(V)$ are $GL(V)$-invariant, and $V \rightsquigarrow S^d(V)$ and $V \rightsquigarrow \Lambda^d(V)$ are finite dimensional submodules of S^* and Λ^*, respectively. It is also clear that the subspace of elements with zero constant term $S_+^*(V) = \bigoplus_{d=1}^{\infty} S^d(V)$ is a submodule.

If A is an arbitrary associative algebra or associative algebra with unit, then any linear map $V \longrightarrow A$ can be uniquely extended to an algebra homomorphism $S_+^*(V) \longrightarrow A$, or a homomorphism of algebras with unit $S^*(V) \longrightarrow A$, respectively.

Now let $k = \mathbf{F}_2$ as earlier. Let $J(V) \subseteq S_+^*(V)$ be the ideal generated by all elements $v^2 - v$ ($= v^2 + v$), where $v \in V$, and let $R(V) = S_+^*(V)/J(V)$ be the quotient algebra. For any $f : V \longrightarrow U$ we have $f_*(v^2 - v) = f(v)^2 - f(v) \in J(U)$. Hence $f_*(J(V)) \subseteq J(U)$, and so f_* induces a homomorphism of quotient algebras $R(f) : R(V) \longrightarrow R(U)$. Thus, R is a functor from $\mathbf{fVect}_k$ to the category of commutative associative algebras. We shall see in the proof of 4.13 that $R(V)$ is finite dimensional.

Define $Q_1 = R|_{\mathcal{V}}$.

4.13. *$Q_1 \cong Q$ as $k\mathcal{V}$-module.*

Proof. Let $X \in \mathcal{V}$. The space $Q(X)$ is an algebra under usual product of functions. Its basis is $\{\delta_l \mid l \in I(X)\}$, where $\delta_l(m) = 0$ or 1 when $m \neq l$ or $m = l$, respectively. Its dimension equals $\dim Q(X) = |I(X)| = |X| - 1 = 2^d - 1$, if $X = U_d$.

An element $v \in X$ induces a function $h_v : X^* \setminus \{0\} \longrightarrow k$ in the usual way, i.e., $h_v(l) = l(v)$. Then $h_u + h_v = h_{u+v}$. We also denote h_v by $\bar{v}$. Let $h : X \longrightarrow Q(X)$ be the linear map taking v to $\bar{v} = h_v$.

Due to the universal property of symmetric algebras h can be uniquely extended to an algebra homomorphism $h_* : S_+^*(X) \longrightarrow Q(X)$. Since $x^2 \equiv x$ in k, it follows that $a^2 = a$ for any $a \in Q(X)$. In particular, $\bar{v}^2 = \bar{v}$ for any $v \in X$. Hence $h_*(J(X)) = 0$, and so h_* induces homomorphism of algebras $\bar{h}_* : Q_1(X) \longrightarrow Q(X)$.

Let $e_1, \dots, e_d$ be a basis of $X = U_d$ and $e^1, \dots, e^d$ be the dual basis of X^*. Let $l \in X^* \setminus \{0\}$, $l = \sum_{i=1}^d l_i e^i$, where $(l_1, \dots, l_d) \neq (0, \dots, 0)$. It is easy to see that $\delta_l = \prod_{i=1}^d (\overline{e_i} + l_i + 1)$. Since $l_i = 1$ for some i, the latter polynomial in $\overline{e_1}, \dots, \overline{e_d}$ has no constant term. So every $a \in Q(X)$ can be represented as a polynomial in $\overline{e_1}, \dots, \overline{e_d}$ without constant term. It follows that $\bar{h}_*$ is surjective.

Since $e_i^2 - e_i \in J(X)$, it follows that $e_i^t - e_i \in J(X)$ for any $t \geqslant 1$. So every $g \in S_+^*(X)$ is congruent modulo $J(X)$ to an element g_1 that has degree $\leqslant 1$ in each e_i (and has no constant term). Hence $\dim Q_1(X) \leqslant 2^d - 1$. Since $\bar{h}_*$ is surjective and $\dim Q(X) = 2^d - 1$, we see that $\bar{h}_*$ is an isomorphism.

Thus we have defined an algebra isomorphism $\bar{h}_* = \bar{h}_{*,X} : Q_1(X) \longrightarrow Q(X)$ for each $X \in \mathcal{V}$. Now we prove that $\bar{h}_* = \{\bar{h}_{*,X} \mid X \in \mathcal{V}\}$ is a $k\mathcal{V}$-module homomorphism, i.e., for any $X, Y \in \mathcal{V}$ and $f \in \mathrm{Hom}_{\mathcal{V}}(X, Y)$ the diagram

$$\begin{tikzcd} Q_1(X) \arrow[r, "Q_1(f)"] \arrow[d, "\bar{h}_{*,X}"'] & Q_1(Y) \arrow[d, "\bar{h}_{*,Y}"] \\ Q(X) \arrow[r, "Q(f)"] & Q(Y) \end{tikzcd}$$

commutes.

The upper and both vertical arrows of this diagram are k-algebra homomorphisms. Next, if $\tau : I \longrightarrow J$ is an arbitrary map of finite sets, then $\tau^* : k^J \longrightarrow k^I$ is an algebra homomorphism (with respect to the usual pointwise multiplication of functions in $k^I = \mathrm{Fun}(I, k)$ and k^J). Therefore $Q(f)$ is an algebra homomorphism also. Now, as $X_1 = (X + J(X))/J(X)$ generates $Q_1(X)$ and since all arrows in the diagram are algebra homomorphisms, it suffices to prove that

$$\bar{h}_{*,Y} Q_1(f)\big(v + J(X)\big) = Q(f)\bar{h}_{*,X}\big(v + J(X)\big) \tag{12}$$

for all $v \in X$.

We have $Q_1(f)(v + J(X)) = f(v) + J(Y)$, $\bar{h}_{*,Y}(f(v) + J(Y)) = \overline{f(v)} = h_{f(v)}$, and $\bar{h}_{*,X}(v + J(X)) = \bar{v} = h_v$. So the equality we need turns into

$$h_{f(v)} = Q(f)h_v. \tag{13}$$

The latter equality means that $h_{f(v)}(l) = (Q(f)h_v)(l)$ for every $l \in Y^* \setminus \{0\}$. But $h_{f(v)}(l) = l(f(v))$ by the definitions, and also $(Q(f)h_v)(l) = h_v(f^*(l)) = (f^*(l))(v) = l(f(v))$, the same element. So the equalities (13) and (12) are proved. Thus, $\bar{h}_*$ is a homomorphism of $k\mathcal{V}$-modules. Since $\bar{h}_{*,X}$ is an isomorphism for each $X \in \mathcal{V}$, it follows from 2.7 that $\bar{h}_*$ is an isomorphism of $k\mathcal{V}$-modules. □

GL(V)-modules In this paragraph $V \in \mathcal{V}$. We study the submodule structure of various $GL(V)$-modules.

Let $d \geqslant 1$ and $0 \leqslant l \leqslant d$. By $C(d, l)$ we denote the set of all subsets of l elements

$$K = \{i_1 < \cdots < i_l\} \subseteq \{1, \ldots, d\}.$$

The array $K_0 = \{1, \ldots, l\}$ is the “least” element of $C(d, l)$.

Let $\dim V = d$ and let $e_1, \ldots, e_d$ be the basis of V. For $K = \{i_1 < \cdots < i_l\} \in C(d, l)$ consider the elements $\widehat{e}_K = e_{i_1} \wedge \cdots \wedge e_{i_l} \in \Lambda^l(V)$. Then $\{\widehat{e}_K \mid K \in C(d, l)\}$ is a basis of $\Lambda^l(V)$.

Let $1 \leqslant i \neq j \leqslant d$. Denote by x_{ij} the transvection from $GL(V)$, i.e., $x_{ij}(e_j) = e_j + e_i$ and $x_{ij}e_l = e_l$ when $l \neq j$. Similarly, let w_{ij} be the transposition on the set of basis vectors: $e_i \leftrightarrow e_j$, $e_l \longmapsto e_l$ when $l \neq i, j$. Note that $GL(V) = SL(V)$ as $k = \mathbf{F}_2$, so the transvections x_{ij} generate $GL(V)$.

Finally, let $U \subseteq GL(V)$ be the subgroup of all transformations g that are triangular with respect to basis $\{e_i\}$ with unity on the diagonal, i.e., $ge_1 = e_1$, $ge_i = e_i + \sum_{j=1}^{i-1} a_{ij}e_j$, $a_{ij} \in k$, for all $2 \leqslant i \leqslant d$. Then U is a 2-group (and even a Sylow 2-subgroup of $GL(V)$, though we do not need this fact).

We include the proof of the following classical statement to make the text self-contained.

4.14. *Let $1 \leqslant l \leqslant d-1$. Then*

1) *if $t \in \Lambda^l(V)$ and $t \wedge e_i = 0$ for all $i = 1, \dots, d$, then $t = 0$;*
2) *the element $\widehat{e}_{K_0} = e_1 \wedge \cdots \wedge e_l$ is the unique nontrivial U-invariant element of $\Lambda^l(V)$;*
3) *$\Lambda^l(V)$ is an irreducible $GL(V)$-module.*

Proof. 1) Consider the decomposition $t = \sum a_K \widehat{e}_K$, the sum is over all $K \in C(d,l)$. Consider arbitrary $K \in C(d,l)$, and let $K' = \{1, \dots, d\} \setminus K$. Then $t \wedge e_{K'} = a_K e_1 \wedge \cdots \wedge e_d$. On the other hand, since $t \wedge e_i = 0$ for all i, it follows that $t \wedge e_{K'} = 0$, whence $a_K = 0$.

2) Let $t \in \Lambda^l(V)$ be a nontrivial element such that $gt = t$ for all $g \in U$. Let $1 \leqslant r \leqslant d$ be maximal such that the expression for t involves e_r. Then $r \geqslant l$, and $r = l$ implies $t = e_1 \wedge \cdots \wedge e_l$. So suppose $r > l$. Then $t = t_1 + t_2 \wedge e_r$, where $t_1 \in \Lambda^l(V')$, $t_2 \in \Lambda^{l-1}(V')$, and $V' = \langle e_1, \dots, e_{r-1}\rangle$, and $t_2 \neq 0$.

Let $1 \leqslant s \leqslant r-1$, then $x_{sr} \in U$, so $x_{sr}t = t$. But $x_{sr}t = t_1 + t_2 \wedge (e_r + e_s)$, whence $t_2 \wedge e_s = 0$. Since $l - 1 < r - 1 = \dim V'$, it follows that 1) implies $t_2 = 0$, a contradiction.

3) Let $M \subseteq \Lambda^l(V)$ be a nontrivial $GL(V)$-submodule. Since U is a 2-subgroup, it follows that there exists a nontrivial element $t \in M$ such that $gt = t$ for all $g \in U$. Then $t = \widehat{e}_{K_0}$ by 2). But each monomial $\widehat{e}_K$, where $K \in C(d,l)$, can be obtained from $\widehat{e}_{K_0}$ by an appropriate permutation of basis vectors. So $\langle GL(V)t\rangle = \Lambda^l(V)$, whence $M = \Lambda^l(V)$. $\square$

Let $V \in \mathbf{fVect}_k$ and $d = \dim V$. Let $S_+^{\leqslant l} \subseteq S_+^*(V)$ be the subspace of all elements of degree $\leqslant l$, and let $L_l = L_l(V)$ be the image of $S_+^{\leqslant l}$ in $Q_1(V)$. It is clear that L_l is a $GL(V)$-submodule of $Q_1(V)$. Thus we have a submodule series

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{d-1} \subseteq L_d = Q_1(V).$$

For $K = \{i_1 < \cdots < i_l\} \in C(d,l)$ let $e_K = e_{i_1} \cdots e_{i_l} \in S^l(V)$, and let $\widetilde{e}_K = e_K + J(V)$ be the image of e_K in $Q_1(V)$. Generally, for an element $x \in S_+^*(V)$ we denote by $\widetilde{x}$ its image in $Q_1(V)$. Observe that $\widetilde{e}_K = \widetilde{e}_{i_1} \cdots \widetilde{e}_{i_l}$. It is easy to see that $\{\widetilde{e}_K \mid K \in C(d,l)\}$ is a basis of $L_l(V)$ modulo $L_{l-1}(V)$. So, the rule $\widetilde{e}_K \leftrightarrow \widehat{e}_K$ defines a bijection of this basis with the basis $\{\widehat{e}_K\}$ of $\Lambda^l(V)$. Let $\tau_l : L_l/L_{l-1} \longrightarrow \Lambda^l(V)$ be the isomorphism of spaces that corresponds to this bijection of bases. It is easy to see that τ_l is an isomorphism of $GL(V)$-modules (it is sufficient to check that τ_l is compatible with the action of transvections). Therefore,

4.15. *$L_l/L_{l-1} \cong \Lambda^l(V)$ is an irreducible $GL(V)$-module.*

4.16. *Let $2 \leqslant l \leqslant d-1$. Then L_{l-1}/L_{l-2} is the unique nontrivial proper submodule of L_l/L_{l-2}.*

Proof. Assume the contrary. Since both L_{l-1}/L_{l-2} and L_l/L_{l-1} are irreducible, it follows that there exists a submodule M such that $L_l/L_{l-2} = M \oplus L_{l-1}/L_{l-2}$. It is easy to see that the composition map

$$f : M \longrightarrow M \oplus L_{l-1}/L_{l-2} = L_l/L_{l-2} \longrightarrow L_l/L_{l-1} \xrightarrow{\tau_l} \Lambda^l(V)$$

is an isomorphism of $GL(V)$-modules. Let $m = f^{-1}(\widehat{e}_{K_0})$. Then $m = \widetilde{e}_{K_0} + t + L_{l-2}$, where $t = \sum_K a_K \widetilde{e}_K$, the sum is over $K \in C(d, l-1)$. Since $\widehat{e}_{K_0}$ is U-invariant, it follows that m must be U-invariant also.

Let r be the maximal number such that t involves e_r. Assume first $r \geqslant l+1$. Then $t = t_1 + t_2\widetilde{e}_r$, where neither t_1 nor t_2 contain $\widetilde{e}_r$, $t_2 \neq 0$, and $\deg t_1 = l-1$ and $\deg t_2 = l-2$, respectively. Take any s such that $1 \leqslant s \leqslant r-1$, then the transvection x_{sr} leaves $\widetilde{e}_1 \cdots \widetilde{e}_l$, t_1, and t_2 invariant, whence $x_{sr}m - m = t_2\widetilde{e}_s + L_{l-2}$. Moreover, $x_{sr}m = m$, because $x_{sr} \in U$. So $t_2\widetilde{e}_s \in L_{l-2}$, hence each monomial $\widetilde{e}_K$ involved in t_2 contains e_s. Since this is true for all $s \leqslant r-1$, we see that $\deg t_2 \geqslant r-1 \geqslant l$, a contradiction. Thus $r \leqslant l$.

Since $r \leqslant l$, it follows that $t = \sum_{i=1}^{l} b_i \widetilde{y}_i$, where $b_i \in k$ and $y_i = e_1 \cdots \widehat{e_i} \cdots e_l$, and $\widehat{\ }$ means that the corresponding factor is omitted. Let $1 \leqslant i \neq j \leqslant l$; then the transvection x_{ij} leaves $\widehat{e}_{K_0}$ invariant and so must fix m.

It is easy to compute that $x_{ij}\widetilde{y}_i = \widetilde{y}_i + \widetilde{y}_j$ (where the equality is modulo L_{l-2}), $x_{ij}(\widetilde{e}_{K_0}) = \widetilde{e}_{K_0} + \widetilde{y}_j$, and $x_{ij}\widetilde{y}_a - \widetilde{y}_a \in L_{l-2}$ when $a \neq i$. Hence $x_{ij}m - m = (1 + b_i)\widetilde{y}_j + L_{l-2}$. Since $x_{ij}m = m$, it follows that $b_i = 1$ for all $1 \leqslant i \leqslant l$, whence

$$m = \widetilde{e}_1 \cdots \widetilde{e}_l + \sum_{i=1}^{l} \widetilde{e}_1 \cdots \widehat{e_i} \cdots \widetilde{e}_l + L_{l-2}.$$

Finally, it is easy to compute that $m + x_{l+1,l}m + w_{l,l+1}m = \widetilde{y}_l + L_{l-2}$. Thus the $GL(V)$-span of m contains a nontrivial element of L_{l-1}/L_{l-2}, a contradiction. $\square$

Consider the element $p = \prod_{i=1}^{d}(1 + \widetilde{e}_i) = \prod_{i=1}^{d}(1 + e_i) + J(V) \in Q_1(V)$, and let $P = \langle p \rangle$ be the one-dimensional space spanned by this element.

4.17. *P is a submodule of $Q_1(V)$, and $Q_1(V) = L_{d-1} \oplus P$.*

Proof. It is obvious that $x_{ij}((1 + \widetilde{e}_l)) = 1 + \widetilde{e}_l$ when $l \neq i, j$ and $x_{ij}((1 + \widetilde{e}_i)(1 + \widetilde{e}_j)) = (1 + \widetilde{e}_i)(1 + \widetilde{e}_i + \widetilde{e}_j) = (1 + \widetilde{e}_i)(1 + \widetilde{e}_j)$, since $\widetilde{e}_i^2 = \widetilde{e}_i$. So $x_{ij}p = p$ for any i an j, whence p is $GL(V)$-invariant.

The equality $Q_1(V) = L_{d-1} \oplus P$ follows since the basis of $L_d = Q_1(V)$ modulo L_{d-1} consists of the unique monomial $\widetilde{e}_1 \cdots \widetilde{e}_d$. $\square$

We leave to the reader to prove the following general fact.

Lemma 4.18. *Let G be a finite group, F be a field, A and B be finite dimensional FG-modules with no common composition factors, and let $M \subseteq A \oplus B$ be a submodule. Then $M = (M \cap A) \oplus (M \cap B)$.*

4.19. *The submodules of $Q_1(V)$ are precisely all L_i and $L_i \oplus P$, where $0 \leqslant i \leqslant d - 1$.*

Proof. By 4.17, $Q_1(V) = L_{d-1} \oplus P$. It follows from 4.15 and 4.14 that the composition factors of L_{d-1} are all $\Lambda^l(V)$, where $1 \leqslant l \leqslant d - 1$. In particular, no of these factors is one-dimensional. So for every submodule $M \subseteq Q_1(V)$ we have $M = (M \cap L_{d-1}) \oplus (M \cap P)$.

It remains to show that the submodules of L_{d-1} are precisely L_i, for all $0 \leqslant i \leqslant d - 1$.

Use induction on $j = 1, \dots, d - 1$ to prove that the submodules of L_j are precisely all L_i with $0 \leqslant i \leqslant j$. For $j = 1$ the statement is obvious. For $j = 2$ the desired statement follows from 4.16.

Let $j \geqslant 3$, and let $M \subseteq L_j$ be a submodule. Let $\overline{M} = M + L_{j-2}/L_{j-2}$ be the image of M in L_j/L_{j-2}. If $\overline{M} \subseteq L_{j-1}/L_{j-2}$, then $M \subseteq L_{j-1}$, whence $M = L_i$ by the induction assumption, where $0 \leqslant i \leqslant j - 1$. If $\overline{M} \nsubseteq L_{j-1}/L_{j-2}$, then $\overline{M} = L_j/L_{j-2}$ by 4.16, whence $\overline{M} \supseteq L_{j-1}/L_{j-2}$. Hence $M \cap L_{j-1} \nsubseteq L_{j-2}$. Again by the induction assumption we find $M \cap L_{j-1} = L_{j-1}$, i.e., $M \supseteq L_{j-1}$. Since L_j/L_{j-1} is irreducible, $M \supseteq L_{j-1}$ and $M \nsubseteq L_{j-1}$, it follows that $M = L_j$. $\square$

Below we denote L_i and P by $L_i(V)$ and $P(V)$, respectively, if we need to specify the space V.

The $k\mathcal{V}$-submodules of Q_1 Now we can describe the $k\mathcal{V}$-submodules of Q_1. Suppose L is such a submodule. Then $L(V)$ is a $GL(V)$-submodule of $Q_1(V)$ for any $V \in \mathcal{V}$. So $L(V) = L_l(V)$ or $L_l(V) \oplus P(V)$, where $0 \leqslant l \leqslant d - 1$ and $d = \dim V$. We denote $L_l(U_d)$, $P(U_d)$, and $L_l(U_d) \oplus P(U_d)$ by $L_l^{(d)}$, $P^{(d)}$, and $L_l^{(d)} \oplus P^{(d)}$, respectively. It is convenient to define $L_l^{(d)} = Q_1(U_d)$ when $l \geqslant d$.

Observe the following. Let U and V be some finite dimensional spaces over the same field, $\dim U \geqslant \dim V \geqslant 1$, and let $f, g : U \longrightarrow V$ be epimorphisms. Then there exists an element $a \in GL(U)$

such that $g = fa$. In particular, if $b \in GL(V)$, then bf is an epimorphism of U onto V, so there exists $a \in GL(U)$ such that $bf = fa$.

Below we write f_* for $Q_1(f)$.

Now let $U, V \in \mathcal{V}$, let $f : U \longrightarrow V$ be an epimorphism, and let $M \subseteq Q_1(U)$ be a $GL(U)$-submodule. Then $f_*M = Q_1(f)M$ is a $GL(V)$-submodule of $Q_1(V)$, and f_*M depends only on M, but does not depend on f. Indeed, let $b \in GL(V)$. Take $a \in GL(U)$ such that $bf = fa$. Then $b_*(f_*M) = (b_*f_*)M = (bf)_*M = (fa)_*M = (f_*a_*)M = f_*(a_*M) = f_*M$; that is, f_*M is $GL(V)$-invariant. Moreover, if f' is another epimorphism of U onto V, then $f' = fa$ for some $a \in GL(U)$, whence $f'_*M = f_*a_*M = f_*M$.

For each space U_d we fix a basis $\{e_l^{(d)} \mid 1 \leqslant l \leqslant d\}$. Let $i \geqslant j$. Consider epimorphisms $f = f_{ij} : U_i \longrightarrow U_j$ defined by $f_{ij}(e_l^{(i)}) = e_l^{(j)}$ when $l \leqslant j$ and $f_{ij}(e_l^{(i)}) = 0$ when $l > j$. It is easy to see that $f_*P^{(i)} = P^{(j)}$ and $f_*L_l^{(i)} = L_l^{(j)}$. It follows that

$$L_l = \{L_l^{(d)} \mid d \geqslant 1\}, \qquad P = \{P^{(d)} \mid d \geqslant 1\}, \quad \text{and} \quad L_l + P = \{L_l^{(d)} + P^{(d)} \mid d \geqslant 1\}$$

are epimorphic submodules of Q_1 for all $l \geqslant 0$. It is easy to see that all these submodules are distinct and different from Q_1. (For example, if $l_1 < l_2$ and $d > l_2$, then $L_{l_1}^{(d)}$ and $L_{l_2}^{(d)}$ are distinct proper subspaces of $Q_1(U_d)$, so $L_{l_1} \neq L_{l_2}$.)

Prove that each proper epimorphic submodule of Q_1 equals one of submodules P, L_l, or $L_l + P$. We have

$$\begin{aligned}
(f_{ij})_*\big(L_l^{(i)}\big) &= L_l^{(j)}, \quad l \leqslant j-1, \\
(f_{ij})_*\big(L_l^{(i)}\big) &= Q_1(U_j), \quad l \geqslant j, \\
(f_{ij})_*\big(L_l^{(i)} \oplus P^{(i)}\big) &= L_l^{(j)} \oplus P^{(j)}, \quad l \leqslant j-2, \\
(f_{ij})_*\big(L_l^{(i)} \oplus P^{(i)}\big) &= Q_1(U_j), \quad l \geqslant j-1.
\end{aligned} \tag{14}$$

Now suppose $L \subset Q_1$ is an epimorphic submodule. If $L(U_1) \neq Q_1(U_1) = P^{(1)}$, then $L(U_1) = 0$. Then formulae (14) imply that $L(U_i) = 0$ for all $i \geqslant 2$, that is, $L = 0$ $(= L_0)$. So we can assume $L(U_1) = Q(U_1)$. Take the least d such that $L(U_d) \neq Q_1(U_d)$. Then $d \geqslant 2$. Since L is epimorphic, it follows that $(f_{d,d-1})_*(L(U_d)) = L(U_{d-1}) = Q_1(U_{d-1})$, whence $L(U_d) = L_{d-1}^{(d)}$ or $L_{d-2}^{(d)} \oplus P^{(d)}$.

Assume $L(U_d) = L_{d-1}^{(d)}$. Then $(f_{i,d})_*(L(U_i)) = L_{d-1}^{(d)}$ for any $i \geqslant d$. Then (14) implies $L(U_i) = L_{d-1}^{(i)}$. So $L(U_i) = L_{d-1}(U_i)$ for all i, i.e., $L = L_{d-1}$.

Similarly, let $L(U_d) = L_{d-2}^{(d)} \oplus P^{(d)}$. Then (14) implies $L(U_i) = L_{d-2}^{(i)} \oplus P^{(i)}$ for each $i \geqslant d$, whence $L = L_{d-2} + P$.

Thus, we have proved the following.

4.20. *The epimorphic submodules of Q_1 are precisely Q_1, L_l, and $L_l + P$ for $l \geqslant 0$, and all these submodules are pairwise distinct.*

The proof of the following statement is left to the reader.

4.21. *All the inclusions of submodules L_l and $L_l + P$ are the following: $L_i \subset L_j$ and $L_i + P \subset L_j + P$ precisely when $i < j$; $L_i \subset L_j + P$ precisely when $i \leqslant j$; $L_i + P$ is never contained in L_j. Moreover, $\bigcup_{i=0}^{\infty} L_i = \bigcup_{i=0}^{\infty}(L_i + P) = Q_1$.*

The proof of Theorem 2.3 We have $S_{k\mathcal{D}}(H) \setminus \{H\} \cong S_{k\mathcal{D}}(\overline{D}) \cong S_{k\mathcal{D}_1}(H_1) \cong S_{k\mathcal{D}_2}(H_2) \cong S_{k\mathcal{V}}(Q) \cong S_{k\mathcal{V}}(Q_1)$ by statement 4.3, Corollaries 4.6, 4.10, and statements 4.12, 4.13. Let $\gamma : S_{k\mathcal{D}}(H) \setminus \{H\} \longrightarrow S_{k\mathcal{V}}(Q_1)$ be an isomorphism.

Set $D_i = \gamma^{-1}(L_i)$, $E_i = \gamma^{-1}(L_i + P)$, and $\overline{D} = \gamma^{-1}(Q_1)$. Then H, D_i, E_i, and $\overline{D}$ are precisely all different epimorphic submodules of H, and the inclusions among D_i, E_i, and $\overline{D}$ are the same as the inclusions among L_i, $L_i + P$ and Q_1. So the first and the third statements of Theorem 2.3 are established. The second statement easily follows from the first and third ones, because for any k, $\mathcal{C}$ and $M \in k\mathcal{C}$-**Mod** the sum operation on $S_{k\mathcal{C}}(M)$ is determined uniquely by the poset structure on $S_{k\mathcal{C}}(M)$.

5. Concluding remarks

1. The author thinks that it is possible to obtain a reasonable characterization, in terms of their structure, of groups G such that $\mathrm{form}(G)$ contains only finitely many subformations. This problem seems to be hard but not hopeless.

2. In the theory of formations some special types of formations are considered, namely the so-called local and Baer-local (or *composition*, in Russian terminology) formations. See [2, Ch. 1, §3], or [3, §§4.3 and 4.4], for the definitions. For every set $\mathfrak{M}$ of groups there exist local and composition formations $\mathrm{lform}(\mathfrak{M})$ and $\mathrm{cform}(\mathfrak{M})$, respectively, *generated* by $\mathfrak{M}$, i.e., the least local, respectively, composition formation, containing $\mathfrak{M}$. There are the following open "finiteness questions" for such formations:

1. Is it true that $\mathrm{lform}(G)$ contains only finitely many local subformations, for each group G? [2, Ch. 1, §6, n. 5];
2. The same question for composition formations [2, Problem 10];
3. Is it true that for each two local formations $\mathfrak{F}_1$ and $\mathfrak{F}_2$ such that $\mathfrak{F}_1 \nsubseteq \mathfrak{F}_2$ there exists an $\mathfrak{F}_2$-critical subformation of $\mathfrak{F}_1$, that is, a local formation $\mathfrak{L} \subseteq \mathfrak{F}_1$ such that $\mathfrak{L} \nsubseteq \mathfrak{F}_2$ but $\mathfrak{L}_1 \subseteq \mathfrak{F}_2$ for every proper local subformation $\mathfrak{L}_1 \subset \mathfrak{L}$? [4, Problem 9.60].

The answers to all these questions are negative.

Theorem 5.1. *Let A be a group of type $2S_5$, V be a faithful $\mathbf{F}_3A$-module, and let $B = V \rtimes A$ (the semidirect product). Then*

1) $\mathrm{lform}(B)$ *contains infinitely many local subformations*;
2) $\mathrm{cform}(B)$ *contains infinitely many composition subformations*;
3) *there exists a local subformation $\mathfrak{F}_1 \subset \mathrm{lform}(B) = \mathfrak{F}_2$ such that $\mathfrak{F}_2$ has no $\mathfrak{F}_1$-critical local subformations.*

It is not difficult (and it needs only several pages) to deduce the latter theorem from 2.5. However, such a proof requires special formation-theoretic arguments that are completely different from the methods of the present article, and for this reason it will be published separately.

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